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Generalization of the Waterman & Truell formula for an elastic medium containing random configurations of cylindrical scatterers

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Propagation of P and SV waves in an elastic solid containing randomly distributed inclusions in a half-space is investigated. The approach is based on a multiple scattering analysis similar to the one proposed by Waterman and Truell for scalar waves. The characteristic equation, the solution of which yields the effective wave numbers of coherent elastic waves, is obtained. Formulae are derived for the effective wave numbers in a dilute random distribution of identical scatterers. They generalize the formula obtained by Waterman and Truell for scalar coherent waves. It is shown that P and SV waves are coupled for non circular cylinders but uncoupled for circular cylinders.

1 Introduction

We consider the problem of elastic wave propagation in heterogeneous solids containing distributions of inhomogeneities. The results have applications in geophysical exploration and ultrasonic evaluation of composite materials or biological tissues. Typically, the inhomogeneities can be hard grains, inclusions, micro-cracks, fibers, pores, or contrast agents, for instance.

We focus attention on the coherent wave propagation, which is the statistical average of the dynamics corresponding to all possible configurations of the scatterers. It is well-known that the coherent motion makes each heterogeneous medium appear as a dissipative homogeneous material, and propagation is governed by a complex effective wave number that is frequency dependent. The real part of the wave number is related to the velocity, and the imaginary part represents the attenuation.

The method developed here starts from an explicit multiple scattering formulation in which the field scattered from any particular scatterer is expressed as a multipole (far-field) expansion. In the case of scalar waves, this is a classical topic with a large literature [1, 2, 3, 4]. In comparison with the numerous studies of the scalar situation, multiple scattering of elastic waves involving both compressional (P) and shear waves (SV) has received relatively little attention. Varadan *et al.* [5] and Yang & Mal [6] have considered this problem using multipole expansions. Their analysis is mainly focused on the low-frequency limit (Rayleigh limit) which predicts dynamic effective mechanical properties of particulate composites that are in agreement with Hashin and Rosen's bounds [7]. The Generalized Self Consistent Method (GSCM) [6] is derived by using a self consistent scheme applied to Waterman and Truell's formula [8]. However, this formula, which is valid for scalar waves, is applied to P and SV waves separately [6], and mode conversions between P and SV waves are therefore neglected. In contrast to the GSCM, the theory developed in [5] takes mode conversions into account, but the equations that involve P and SV waves are uncoupled by invoking additional hypotheses above and beyond those for the QCA used in most of the papers previously cited. This is not necessary and it can be avoided by ensuring that the equations faithfully and accurately describe the coupling between the P and SV waves, as done in this paper. More precisely, we derive equations for P and SV waves in the spirit of the paper by Waterman and Truell [8], and effective wave numbers are then obtained in the limit as the scatterer size tends to zero. From this point of view, our results can be considered as a generalization of the work of Waterman and Truell for scalar waves to the elastic case. As the present approach is based on multipole expansions, scatterers with non-circular shapes can be handle. The T-matrix method provides a convenient tool to calculate such multipole expansions [9].

2 Multiple scattering formulation

Suppose that time-harmonic P or SV waves are propagating perpendicular to N parallel cylinders located in an elastic solid and that k_L and k_T are the wave numbers of the P and SV waves. We assume the Helmholtz decomposition of the displacement in the form

$$\vec{u} = \vec{\nabla}\psi^L + \vec{\nabla} \times (\psi^T \vec{e_z}) \tag{1}$$

where $\vec{e_z}$ is the unit vector parallel to the cylinders and ψ^L and ψ^T represent potentials for the longitudinal (P) and transverse (SV) components of the waves. Under the influence of the incident waves $\psi^L_{inc}(\vec{r}) = exp(ik_L x)$ and $\psi^T_{inc}(\vec{r}) = exp(ik_T x)$, both L and T scattered waves

$\psi_S^L(\vec{r};\vec{r}_k)$ and $\psi_S^T(\vec{r};\vec{r}_k)$ are generated by the kth scatterer, so that

$$\begin{pmatrix}\psi^L(\vec{r})\\ \psi^T(\vec{r})\end{pmatrix} = \begin{pmatrix}\psi^L_{inc}(\vec{r})\\ \psi^T_{inc}(\vec{r})\end{pmatrix} + \sum_{k=1}^{N}\begin{pmatrix}\psi^L_S(\vec{r};\vec{r}_k)\\ \psi^T_S(\vec{r};\vec{r}_k)\end{pmatrix}. \quad (2)$$

Here, the first vector argument $\vec{r}=(x,y)$ specifies the observation point, while $\vec{r}_k$ is the location of the kth scatterer. The scatters are assumed to be identical in composition and orientation and the properties of a single scatterer are assumed to be known, so that a rule is available that relates the scattered waves $\psi_S^\alpha(\vec{r};\vec{r}_k)$ and the exciting fields $\psi_E^\alpha(\vec{r};\vec{r}_k)$ acting on the kth scatterer ($\alpha = L,T$). This rule defines a linear scattering operator $T(\vec{r}_k)$, with components $T^{\alpha\beta}(\vec{r}_k)$ ($\alpha,\beta\in\{L,T\}$), by the relations [9]

$$\begin{pmatrix}\psi^L_S(\vec{r};\vec{r}_k)\\ \psi^T_S(\vec{r};\vec{r}_k)\end{pmatrix} = \begin{bmatrix} T^{LL}(\vec{r}_k) & T^{TL}(\vec{r}_k)\\ T^{LT}(\vec{r}_k) & T^{TT}(\vec{r}_k)\end{bmatrix}\begin{pmatrix}\psi^L_E(\vec{r};\vec{r}_k)\\ \psi^T_E(\vec{r};\vec{r}_k)\end{pmatrix}.$$

The exciting field acting on the kth scatterer is the sum of the incident waves and the scattered waves from all scatterers other than the kth. It follows that

$$\begin{pmatrix}\psi^L_E(\vec{r};\vec{r}_k)\\ \psi^T_E(\vec{r};\vec{r}_k)\end{pmatrix} = \begin{pmatrix}\psi^L_{inc}(\vec{r})\\ \psi^T_{inc}(\vec{r})\end{pmatrix}$$
$$+\sum_{j\neq k}\begin{bmatrix} T^{LL}(\vec{r}_j) & T^{TL}(\vec{r}_j)\\ T^{LT}(\vec{r}_j) & T^{TT}(\vec{r}_j)\end{bmatrix}\begin{pmatrix}\psi^L_E(\vec{r};\vec{r}_j)\\ \psi^T_E(\vec{r};\vec{r}_j)\end{pmatrix}. \quad (3)$$

Equations (2) to (3) are the multiple scattering equations that generalize those obtained for the scalar case (*cf.* eqs. (2.9-10) in [8]).

3 Modal equations

In order to derive the equations governing the coherent motion, we use the method initially developed by Foldy [11] to average over all possible configurations of cylinders. This method is very well documented [2], and it includes as a special case the quasi-crystalline approximation (QCA). Performing the configurational average transforms eq. (3) into

$$\begin{pmatrix}\langle\psi^L_E(\vec{r};\vec{r}_1)\rangle\\ \langle\psi^T_E(\vec{r};\vec{r}_1)\rangle\end{pmatrix} = \begin{pmatrix}\psi^L_{inc}(\vec{r})\\ \psi^T_{inc}(\vec{r})\end{pmatrix} + \int d\vec{r}_j\, n(\vec{r}_j,\vec{r}_1)$$
$$\begin{bmatrix} T^{LL}(\vec{r}_j) & T^{TL}(\vec{r}_j)\\ T^{LT}(\vec{r}_j) & T^{TT}(\vec{r}_j)\end{bmatrix}\begin{pmatrix}\langle\psi^L_E(\vec{r};\vec{r}_j)\rangle\\ \langle\psi^T_E(\vec{r};\vec{r}_j)\rangle\end{pmatrix}. \quad (4)$$

In these equations, $\vec{r}_1$ is the location of one of the cylinders, $\langle\psi_E^\alpha(\vec{r};\vec{r}_j)\rangle$ ($\alpha\in\{L,T\}$) are the average coherent fields acting on the jth scatterer, $n(\vec{r},\vec{r}_j)$ the conditional number density of scatterers at $\vec{r}$ if a scatterer is known to be at $\vec{r}_j$, and the integral is taken over the whole surface accessible to scatterers.

In the same way as $\psi_E^{L,T}(\vec{r};\vec{r}_j)$, the effective potentials $\langle\psi_E^{L,T}(\vec{r};\vec{r}_j)\rangle$ satisfy the Helmholtz equation and are regular functions at the point $\vec{r_j}$, they can therefore be expressed ($\alpha\in\{L,T\}$)

$$\langle\psi_E^\alpha(\vec{r};\vec{r}_j)\rangle = \sum_n A_n^\alpha(\vec{r}_j)\, J_n(k_\alpha\rho_j)e^{in\theta(\vec{\rho}_j)}, \quad (5)$$

with $\vec{\rho}_j=\vec{r}-\vec{r}_j$, $\theta(\vec{\rho}_j)=\arg(\vec{\rho}_j)$ and $\rho_j=|\vec{\rho}_j|$. As usual with the T-matrix approach [9], the transition operators are defined by ($\alpha,\beta\in\{L,T\}$)

$$T^{\alpha\beta}(\vec{r}_j)J_n(k_\alpha\rho_j)e^{in\theta(\vec{\rho}_j)} = T_n^{\alpha\beta}H_n^{(1)}(k_\alpha\rho_j)e^{in\theta(\vec{\rho}_j)}, \quad (6)$$

and the corresponding far-field scattering amplitudes of the different interactions are given by

$$T^{\alpha\beta}(\vec{0})e^{ik_\alpha x} = \sum_n T_n^{\alpha\beta}H_n^{(1)}(k_\alpha r)e^{in\theta} \quad (7)$$
$$\simeq \sqrt{\frac{2}{\pi k_\alpha r}}e^{i(k_\alpha r-\frac{\pi}{4})}\, f^{\alpha\beta}(\theta) \quad (r\to\infty)$$

with $\vec{r}=(r\cos\theta, r\sin\theta)$. The far-field scattering functions $f^{\alpha\beta}(\theta)$ are therefore Fourier series with coefficients equal to the modal scattering amplitudes $T_n^{\alpha\beta}$, *i.e.* ($\alpha,\beta\in\{L,T\}$)

$$f^{\alpha\beta}(\theta) = \sum_n T_n^{\alpha\beta}e^{in\theta}. \quad (8)$$

Modal coefficients $T_n^{\alpha\beta}$ can be calculated numerically [9, 10]. For circular cylinders, they are the components of the T-matrix and satisfy the symmetry relation $T_{-n}^{\alpha\beta} = T_n^{\alpha\beta}$. For non circular cylinders, they are expressed in terms of the T-matrix components that depend on the orientation of the scatterer, so that in general $T_{-n}^{\alpha\beta}\neq T_n^{\alpha\beta}$.

As we seek coherent waves that propagate in the equivalent homogeneous medium, we assume that the solutions of eqs. (4), taking eqs. (5-6) into account, may be written in the form

$$\begin{pmatrix}A_n^L(\vec{r}_j)\\ A_n^T(\vec{r}_j)\end{pmatrix} = i^n\begin{bmatrix}A_n^L & B_n^L\\ A_n^T & B_n^T\end{bmatrix}\begin{pmatrix}e^{i\xi x_j}\\ e^{i\xi' x_j}\end{pmatrix}. \quad (9)$$

Here ξ and ξ' are the effective wave numbers of coherent waves that propagate in the direction of the $x-$axis, and the coefficients A_n^L and B_n^L are at this stage unknown. Here, two coherent waves with ξ and ξ' as wave numbers are assumed to propagate, which is a natural hypothesis for scarce concentrations of scatterers. In such situations the homogeneous medium looks like an elastic medium in which the two waves that propagate are predominantly P or SV waves.

The Waterman and Truell approximation assumes a pair correlation function with the following property [2, 8]

$$n(\vec{r},\vec{r}_j) = n_0 \quad \text{for} \quad |x-x_j|>\eta \quad (10)$$

$$n(\vec{r},\vec{r}_j) = 0 \quad \text{for} \quad |x-x_j|<\eta \quad (11)$$

for $\eta\to 0$ with $\vec{r}=(x,y)$ and $\vec{r}_j=(x_j,y_j)$. In this limit eqs. (4) are improper integrals in the sense of Cauchy principal value, which can be calculated as in [8]. Having used the Lorentz-Lorenz law, we find

$$A_n^L - \frac{2n_0}{ik_L}\sum_p\left(T_p^{LL}A_p^L + T_p^{TL}A_p^T\right)\left[\frac{1}{\xi-k_L} - \frac{(-1)^{n+p}}{\xi+k_L}\right] = 0, \quad (12)$$

$$A_n^T - \frac{2n_0}{ik_T}\sum_p\left(T_p^{TT}A_p^T + T_p^{LT}A_p^L\right)\left[\frac{1}{\xi-k_T} - \frac{(-1)^{n+p}}{\xi+k_T}\right] = 0, \quad (13)$$

and

$$B_n^L - \frac{2n_0}{ik_L}\sum_p \big(T_p^{LL}B_p^L + T_p^{TL}B_p^T\big)\big[\frac{1}{\xi' - k_L} - \frac{(-1)^{n+p}}{\xi' + k_L}\big] = 0, \tag{14}$$

$$B_n^T - \frac{2n_0}{ik_T}\sum_p \big(T_p^{TT}B_p^T + T_p^{LT}B_p^L\big)\big[\frac{1}{\xi' - k_T} - \frac{(-1)^{n+p}}{\xi' + k_T}\big] = 0. \tag{15}$$

Eqs. (12,13) and eqs. (14,15) provide two identical homogeneous systems of linear algebraic equations which involve either the unknowns $\{A_p^L, A_p^T\}$ with ξ or $\{B_p^L, B_p^T\}$ with $\xi^{'}$. The existence of nontrivial solutions of the homogeneous system determines the effective wave numbers ξ and $\xi^{'}$. These are looked for in the next section.

4 The Waterman & Truell formula for an elastic medium

Let us consider the coupled systems of infinite equations eqs. (12,13), for example, it can still be written

$$A_n^L + P_{LT} + (-1)^n Q_{LT} = 0, \tag{16}$$

$$A_n^T + P_{TL} + (-1)^n Q_{TL} = 0, \tag{17}$$

with

$$\begin{aligned} P_{LT} &= P_L \sum_p \big(T_p^{LL}A_p^L + T_p^{TL}A_p^T\big), \\ Q_{LT} &= Q_L \sum_p (-1)^p \big(T_p^{LL}A_p^L + T_p^{TL}A_p^T\big), \\ P_{TL} &= P_T \sum_p \big(T_p^{TT}A_p^T + T_p^{LT}A_p^L\big), \\ Q_{TL} &= Q_T \sum_p (-1)^p \big(T_p^{TT}A_p^T + T_p^{LT}A_p^L\big), \end{aligned} \tag{18}$$

and ($\alpha = \{L, T\}$)

$$P_\alpha = \frac{2n_0}{ik_\alpha}\frac{1}{k_\alpha - \xi}, \qquad Q_\alpha = \frac{2n_0}{ik_\alpha}\frac{1}{k_\alpha + \xi}. \tag{19}$$

The structure of eqs. (16,17) implies the identities

$$A_{-n}^{L,T} = A_n^{L,T} \quad \text{and} \quad A_{n+2}^{L,T} = A_n^{L,T}. \tag{20}$$

Consequently, the problem reduces from calculating an infinite set of unknowns to one with eight unknowns : P_{LT}, P_{TL}, Q_{LT}, Q_{TL} and $A_{0,1}^{L,T}$ satisfying a system of eight homogeneous linear equations. Note that although $P_{LT}, \ldots, Q_{TL}$ can be expressed in terms of $A_{0,1}^{L,T}$, the calculations are simpler with $P_{LT}, \ldots, Q_{TL}$ considered as unknowns. The first four equations are obtained by setting $n = 0$ and $n = 1$ in eqs. (16,17). Then, we perform an iteration on eqs. (16,17) using the identities eqs. (20), with the result

$$\begin{aligned} A_n^L &- \big[f^{LL}(0)P_L + (-1)^n f^{LL}(\pi)Q_L\big]P_{LT} \\ &- \big[f^{LL}(\pi)P_L + (-1)^n f^{LL}(0)Q_L\big]Q_{LT} \\ &- \big[f^{TL}(0)P_L + (-1)^n f^{TL}(\pi)Q_L\big]P_{TL} \\ &- \big[f^{TL}(\pi)P_L + (-1)^n f^{TL}(0)Q_L\big]Q_{TL} = 0, \end{aligned} \tag{21}$$

$$\begin{aligned} A_n^T &- \big[f^{TT}(0)P_T + (-1)^n f^{TT}(\pi)Q_T\big]P_{TL} \\ &- \big[f^{TT}(\pi)P_T + (-1)^n f^{TT}(0)Q_T\big]Q_{TL} \\ &- \big[f^{LT}(0)P_T + (-1)^n f^{LT}(\pi)Q_T\big]P_{LT} \\ &- \big[f^{LT}(\pi)P_T + (-1)^n f^{LT}(0)Q_T\big]Q_{LT} = 0. \end{aligned} \tag{22}$$

Four equations are obtained from eqs. (21,22) by considering the two possibilities for $(-1)^n$, corresponding to $n = 0$ and $n = 1$. The second set of four equations follow from eqs. (18) combined with the identities eqs. (20). Eliminating $A_{0,1}^{L,T}$ from the eight equations results in the following four equations for the four unknowns P_{LT}, P_{TL}, Q_{LT}, Q_{TL},

$$\begin{aligned} &\big[1 + f^{LL}(0)P_L \pm f^{LL}(\pi)Q_L\big]P_{LT} \\ \pm&\big[1 \pm f^{LL}(\pi)P_L + f^{LL}(0)Q_L\big]Q_{LT} \\ +&\big[f^{TL}(0)P_L \pm f^{TL}(\pi)Q_L\big]P_{TL} \\ +&\big[f^{TL}(\pi)P_L \pm f^{TL}(0)Q_L\big]Q_{TL} = 0, \end{aligned} \tag{23}$$

$$\begin{aligned} &\big[1 + f^{TT}(0)P_T \pm f^{TT}(\pi)Q_T\big]P_{TL} \\ \pm&\big[1 \pm f^{TT}(\pi)P_T + f^{TT}(0)Q_T\big]Q_{TL} \\ +&\big[f^{LT}(0)P_T \pm f^{LT}(\pi)Q_T\big]P_{LT} \\ +&\big[f^{LT}(\pi)P_T \pm f^{LT}(0)Q_T\big]Q_{LT} = 0. \end{aligned} \tag{24}$$

The homogeneous linear system of equations eqs. (23,24) has nontrivial solutions if the associated determinant vanishes. Thus, the modal equation is

$$\det \begin{bmatrix} A_{LL} & A_{TL} \\ A_{LT} & A_{TT} \end{bmatrix} = 0. \tag{25}$$

with

$$A_{LL} = \begin{pmatrix} 1 + f^{LL}(0)P_L & f^{LL}(\pi)P_L \\ f^{LL}(\pi)Q_L & 1 + f^{LL}(0)Q_L \end{pmatrix}, \tag{26}$$

$$A_{TT} = \begin{pmatrix} 1 + f^{TT}(0)P_T & f^{TT}(\pi)P_T \\ f^{TT}(\pi)Q_T & 1 + f^{TT}(0)Q_T \end{pmatrix}, \tag{27}$$

$$A_{TL} = \begin{pmatrix} f^{TL}(0)P_L & f^{TL}(\pi)P_L \\ f^{TL}(\pi)Q_L & f^{TL}(0)Q_L \end{pmatrix}, \tag{28}$$

and

$$A_{LT} = \begin{pmatrix} f^{LT}(0)P_T & f^{LT}(\pi)P_T \\ f^{LT}(\pi)Q_T & f^{LT}(0)Q_T \end{pmatrix}. \tag{29}$$

The formula given by eq. (25) generalizes the identity of Waterman & Truell for acoustic waves in the case of cylindrical coordinates. Equation (25) is a bi-squared equation which has ξ and $\xi^{'}$ for solutions.

5 Formulae for circular cylinders

The mode converted forward scattering and backscattering amplitudes, $f^{LT}(0)$, $f^{TL}(0)$ and $f^{LT}(\pi)$, $f^{TL}(\pi)$ respectively, are identically zero if the fundamental scatterer has sufficient geometrical symmetry. This is the case for circular cylinders, and occurs generally for cylinders with reflection symmetry about the $x-$axis. When $f^{LT}(0) = f^{TL}(0) = f^{LT}(\pi) = f^{TL}(\pi) = 0$, instead of eq. (25), the condition for satisfaction of the four equations eqs. (23,24) becomes two simpler equations

$$\left[1 + f^{LL}(0)P_L\right]\left[1 + f^{LL}(0)Q_L\right] - \left[f^{LL}(\pi)\right]^2 P_L Q_L = 0 \tag{30}$$

and

$$\left[1 + f^{TT}(0)P_T\right]\left[1 + f^{TT}(0)Q_T\right] - \left[f^{TT}(\pi)\right]^2 P_T Q_T = 0. \tag{31}$$

These provide uncoupled modal equations for the P and SV waves. Substituting eqs. (19) into eqs. (30,31), we obtain the well known Waterman & Truell formulae

$$\xi^2 = \left[k_L - \frac{2in_0}{k_L} f^{LL}(0)\right]^2 - \left[\frac{2in_0}{k_L} f^{LL}(\pi)\right]^2 \tag{32}$$

and

$$(\xi')^2 = \left[k_T - \frac{2in_0}{k_T} f^{TT}(0)\right]^2 - \left[\frac{2in_0}{k_T} f^{TT}(\pi)\right]^2 \tag{33}$$

associated to P and SV waves respectively.

6 Other results

Similar formulae have been obtained for poro-elastic media obeying Biot's theory [12]. Three coherent waves may propagate in such media : a shear wave, a fast longitudinal wave, and a slow longitudinal one. While the shear coherent wave propagates uncoupled, the two longitudinal waves are coupled via the scattering by each single circular cylinder. Using Twersky's theory [13], which is less general than the one of Waterman & Truell, the same dispersion equation was found for the shear wave (*cf.* eqs. (31,33)), and a bi-squared coupled one for the fast and slow waves, bi-squared equation which is similar to eqs. (25).

7 Conclusion

Formulae have been derived for the effective wave numbers in a dilute random distribution of identical scatterers embedded in a solid. They generalize the formula obtained by Waterman and Truell for scalar coherent waves. It is shown that P and SV coherent waves are coupled for non circular cylinders but uncoupled for circular cylinders.

However, this result depends on the theory which is used. It has been shown that P and SV coherent waves can be coupled even for circular cylinders when using a more general theory based on the Fikioris & Waterman approach [14].

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