Dispersion des ondes acoustiques transverses horizontales en sub-surface de milieux granulaires non consolides

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Résumé

Lorsqu’ils sont soumis à la pesanteur, les milieux granulaires non consolidés ont la propriété fondamentale, à proximité de leur surface libre, de posséder une variation en loi de puissance des modules élastiques avec la pression ou la profondeur y (en l’absence de variation importante de la densité). Dans cette communication, les prédicions théoriques de la dispersion de la vitesse des ondes acoustiques transverses horizontales (TH) se propagant dans la sub-surface sont présentées. Le guide d’onde élastique de sub-surface est formé par l’augmentation de la rigidité du matériau avec la profondeur ce qui permet de localiser les rayons acoustiques entre la surface libre du matériau granulaire et une caustique située à une profondeur finie de cette surface.

La dépendance en fréquence et en pression de la vitesse de propagation de chaque mode TH guidé est trouvée analytiquement dans deux cas : dans l’approximation géométrique (pour 0<\(\alpha\)=1/2 adapté au cas du milieu granulaire non consolidé) et par résolution de l’équation d’onde du milieu stratifié verticalement dans le cas important où \(\alpha\)=1/4. Les profils des modes TH en fonction de la profondeur (modes localisés en sub-surface et se propageant le long de la surface) sont trouvés analytiquement pour \(\alpha\)=1/4 et sont obtenus numériquement pour 0<\(\alpha\)<1/2.

La théorie développée établit les bases pour une meilleure compréhension de la propagation acoustique à proximité de la surface libre d’un matériau granulaire, en vue d’applications au diagnostic des processus physiques se développant dans cette région, comme les avalanches.

Introduction

The elementary acoustic excitations localized near the surface of the materials are known to be important both for understanding the fundamental physical properties of the material surfaces (for instance, such as heat capacity and thermal conductivity) and for different applications, including nondestructive testing and signal processing. However, despite centuries long lasting interest to near surface phenomena in unconsolidated granular media (for example, such as avalanches and dunes) and extensive research in the area of bulk acoustic modes propagation in these materials (in the increasingly complicated situations accounting for medium micro-inhomogeneity, the existence of force chains, and even the possible jamming, fragility [1-3] and the intrinsic nonlinearity of these materials [1, 4]), the acoustic surface and waveguide modes are nearly unexplored.

The fundamental property of the unconsolidated granular materials under gravity in the vicinity of their mechanically free horizontal surface is a power law variation of their elastic moduli with pressure \(p\) and, consequently, with depth coordinate \(y\) when \(p=\rho gy\) holds near the surface [5]. Here \(\rho\) and \(g\) denote the nearly constant material density and the gravity acceleration, respectively. Although the Hertz theory predicts that corresponding variation in the velocity of both longitudinal (L) and shear (S) bulk waves should increase as \(c_{L,S}=\gamma_{L,S}(\rho gy)^{\alpha}\) with the power \(\alpha=1/6\) [1,5], there are multiple experiments documenting the powers different from 1/6, the most frequent observation being \(\alpha=1/4\). Here \(\gamma_{L,S}\) parameters here depend, in particular, on the elastic properties of the individual grains. The experimental results and plausible theoretical explanations for different power-law dependencies of sound velocities on pressure can be found in [5, 6] and the recent publications [1, 3]. The idea that the vertical stratification of a granular material leads to the upward bending of the acoustic rays (“mirage” effect [7]) is fundamental for the physics of the localized acoustic modes. The theory for waves in inhomogeneous media [8] predicts complete reflection of the acoustic waves from the power-law velocity profiles with \(\alpha<1\) even for the normal wave incidence on the surface with vanishing rigidity. This case covers all the profiles reported until now for granular materials. Only for \(\alpha>1\) the reflected wave is absent for the non-absorbing material in the geometrical acoustic approximation, since the incident wave propagation time towards the surface becomes infinite (acoustic “black hole” [9]).

As soon as the existence of a non-vanishing reflection of the acoustic waves from the top surface of the granular material is understood, it becomes obvious that acoustic rays could be guided by the free surface and the surfaces of their reflection from the increasingly rigid material (the caustics [8]).

Localized modes polarized in the horizontal plane

The Helmholtz equation describing the propagation of the linear shear acoustic waves with the propagation vector \(\hat{k}\) in the sagittal \((x,y)\) plane and polarized along the \(z\)-axis (SH waves, Figure 1) for the shear velocity stratification described by \(c_{s}=c_{s0}(y/y_0)^{\alpha}\) is

\[
(\xi^{2\alpha}u_y)’ + (\Omega^2 - \xi^{2\alpha}q^2)u_z = 0
\]  

(1)

Here the normalized vertical coordinate \(\xi=y/y_0\) the cyclic frequency \(\Omega=\omega\sqrt{\rho g}/c_0\) and the wave vector projection on the horizontal \(x\)-axis \(q'=kx_0\) are introduced. The prime denotes the derivative over the vertical coordinate. The mechanical displacement field can be obtained by multiplying the
solution of Eq. (1) by exp[i(ωt−kxx)], where \( k_x \) is the propagation constant of the waveguide acoustic mode when the field is localized in depth. It should be mentioned that neither the characteristic scale of the wave guide velocity \( c_{y0} \) nor the characteristic depth \( y_0 \) exist in the ideal limiting case of the non-truncated power-law velocity distribution starting from zero value at the surface and infinitely increasing with depth. In the latter case the dependence of the localized wave frequency and velocity on its wave number (Ω−q−1−α) can be obtained from simple scaling (dimensional) analysis [8], but in order to find the proportionality constants and acoustic wave shape functions describing in-depth wave profiles, the Eq. (1) should be solved. Since the solution is self-similar, the proportionality constant depends not on \( c_{y0} \) and \( y_0 \) separately but on their wave-number-independent combination \( c_{ys}y_0^{\alpha} = \gamma(p_0)^\alpha \), which can be calculated theoretically or estimated using experimental data. The normalization introduced in Eq. (1) however suits well for the analysis of the cases where the characteristic scale of the velocity is introduced in the model by surface loading or/and internal compression of granular material due to the adhesion between grains. The characteristic scale can also appear due to saturation of the rigidity depth dependence.

![Figure 1: dependence of the normalized phase velocity \( c_{ys}/c_0 \) on the normalized wave number \( q \) for the SH modes, for \( \alpha = 1/4 \) and \( \alpha = 1/6 \). The insets illustrate the long wave (LW) \( q < 1 \) and the short wave (SW) \( q > 1 \) limits. The upper right inset illustrates schematically the geometry of the problem.](image)

We first applied to Eq. (1) the geometrical acoustics (GA) method (Wentzel, Kramers, Brillouin approximation) [10], in which both the amplitude and the phase (eikonal) of \( u_e \) are assumed to be slowly varying at the scale of \( k_e^{-1} \), the latter being itself depth-dependent. For the acoustic rays propagating from the top surface, the eikonal equation (\( S' \)^2=Ω^2/2\( k_e^{-2} \)) yields the turning points (simple caustics) at \( \xi = (\Omega q)^{1/\alpha} \). However, for the power-law profile with \( \alpha < 1 \) the GA validity conditions fail not only as usually in the vicinity of caustic [10], but also in the vicinity of the free surface due to the singularity of the coefficients of the wave equation at \( y = 0 \). So, when applying resonance (quantization) condition of Bohr-Zommerfeld [9],

\[
2 \int_{y_0}^{\xi_0} dS + \Delta \varphi_c + \Delta \varphi_o = 2\pi(n - 1)
\]
to derive the dispersion relation for the waves guided between the surface and the caustic, it is necessary to take into account not only the phase loss \( \Delta \varphi \) that equals to \( -\pi/2 \) [10] for the wave reflection from a simple caustic, but also the phase loss \( \Delta \varphi_o = -\pi(1-\alpha)/2 \) due to the reflection from the gradient of the material rigidity \( \rho(c_0^2) \) at \( y = 0 \). Note that this gradient is infinite for \( \alpha = 1/2 \) assumed hereupon. The first term in the above resonance condition accounting for the geometrical phase increment is also evaluated analytically. Finally, for the dispersion relation for the SH waves localized near the surface we have

\[
\Omega = \left( \pi \left[ 2n - 1 - \frac{2\alpha - 1}{2\alpha - 2} \frac{\alpha^2}{\Gamma(1-\alpha)\Gamma(1+\alpha)} \right] \right)^{1/\alpha} q^{-1-\alpha}
\]

where \( \Gamma \) denotes the Gamma function, and the integer \( n \) satisfies the inequality \( n(\alpha/3/4)/(\alpha-1) \). The normal modes are numbered in the order of increasing phase velocity and, consequently, increasingly deep penetration into the media. It is also straightforward to obtain in the GA approximation the dispersion relations for the SH modes in the case when the pressure at the surface is nonzero \( p(y=0)=p_0>0 \), due to grains adhesion or external vertical loading by an absolutely rigid frictionless plate. In this case \( \Delta \varphi_o = 0 \), which is well known for the reflection from the mechanically free surface with non-vanishing elastic moduli. The non-vanishing pressure at the surface introduces in the considered physical system both the spatial scale \( y_0 = p_0(p_0) \) and the scale for the acoustic velocity \( c_0 = c_{ys}(y=0) = \gamma(p_0) \). The depth profile of the shear acoustic velocity can be described by \( c_0 = c_{ys}(1+y/y_0)^\alpha \). Accordingly, everywhere in Eq. (1) \( \xi_0 \) should be replaced by \( \xi = 1 + \xi_0 \). This has been taken into account when evaluating the geometrical phase increment and finally the pressure-dependent dispersion relations:

\[
\frac{1}{\alpha} \left( \frac{c_{ys}}{c_0} \right)^{\alpha/2} \Gamma\left[\frac{1}{2}(1+\alpha)\right] \Gamma\left(\frac{1}{2}(2-\alpha)\right) \right]^{1/\alpha} q^{1-\alpha}
\]

\[
- \frac{2}{2\alpha - 1} \frac{1}{\alpha} \left( \frac{c_{ys}}{c_0} \right)^{\alpha} F \left[ \frac{-\frac{1}{2}, \frac{1}{2}(1-\alpha)_-}{\frac{1}{\alpha} - 1} ; \frac{1}{2}, \frac{1}{\alpha} - 1 ; \left( \frac{c_{ys}}{c_0} \right)^2 \right]
\]

\[
= 2\pi(n-3/4)/q
\]

Here \( F \) denotes Gauss hypergeometric function. The dependence of the normalized phase velocity \( c_{ys}/c_0 \) on the normalized wave number \( q \) of the SH modes given implicitly by this solution is presented in Figure 1 for \( \alpha = 1/4 \) and \( \alpha = 1/6 \). For a long acoustic wave with \( q < 1 \), the phase velocity increases infinitely in the absence of saturation of material moduli growth with the depth, and the self-similar asymptotic behavior \( c_{ys}/c_0 \sim q^{-\alpha} \) shows up. Short acoustic waves with \( q > 1 \) feel the perturbation (truncation) of the power-law shear velocity profile by the pressure \( p_0 \). In log-log plot in Figure 1 (left inset) this manifests as a deviation of the curves from the straight lines presenting the self-similar limit. The velocity of the guided mode approaches the characteristic velocity \( c_0 \) also following the power law \( c_{ys}/c_0 \sim (3\pi\alpha)^{1/2} \left[ \log(n-3/4) \right]^{1/2} \). This is illustrated in the right inset in Figure 1. The dependence of the SH localized
modes on pressure \( p_0 \) arises in the obtained solution due to the pressure dependent values \( c_0 = \gamma S p_0 \alpha \) and \( y_0 = p_0 / (\rho g) \). The inset shows the eigenvalues \( \lambda_n \) for the SH modes as functions of the wave number \( q \) in the case of nonzero adhesion \( (p_0 \neq 0) \).

The accuracy of the GA approximation is known to increase for higher order modes with increasing \( n \) and to be the worse for the lowest mode \( n=1 \) [10]. Thus, for the quantitative analysis of the lowest modes, the analytical solutions of Eq. (1) are highly desirable. Unexpectedly we have managed to find the exact analytical solution of Eq. (1) in the important for the granular materials case \( \alpha=1/4 \) by transforming Eq. (1) into the Kummer’s equation for the confluent hypergeometric functions. This previously unknown general analytical solution is

\[
u_y = e^{-\eta} \left[ c_1 U(a, b, 2\eta) + c_2 M(a, b, 2\eta) \right]
\]

where \( U \) and \( M \) are the Kummer functions, \( \eta = (q \xi)^{1/2} \), \( d^{1/2}/2 \), \( a=1-d/2 \), \( b=1/2 \), and \( d=\Omega^2 q^2 \). It is valid in the general case when \( p(y=0) \neq 0 \). However, neither \( U \) nor \( M \) can satisfy both stress-free boundary condition for shear stress at and localization condition \( u_y(y \rightarrow \infty) \rightarrow 0 \). We found the required particular solution by combining the Kummer’s functions as follows:

\[
u_y = ce^{-\eta} \left[ 2\pi^{1/2} \frac{M(a, b, 2\eta)}{\Gamma(a+1/2)} \right] \frac{U(a_n, b, 2\eta)}{U(a_n, b, 2\eta)} \text{ for } 0 \leq \xi \leq \xi_n, \quad (2)
\]

Here \( a,=1-d/2 \). In the acoustic field Eq.(2) both the displacements and the stresses are continuous at the internal surface localized at the depth \( \xi_n = d_n/(4q) \). The values \( d_n \) of the parameter \( d \) corresponding to SH localized modes of different order \( n \), are obtained through the solution of the equation

\[
\pi^2 \frac{1}{\Gamma(a_n+1, b+1, 2\eta_n)} M(a_n, b, 2\eta_n) - a_n U(a_n+1, b+1, 2\eta_n) = 0
\]

which represents the condition of the shear stress absence at the surface \( y=0 \), where \( \eta = \eta_n = q^{1/2} \xi_n^{1/2} \). As it has been expected, the eigenvalues \( \lambda_n \) obtained through the solution of Eq. (3) in the limiting case \( p_0=0 \) notably differ from the prediction \( a_n^{(\mathrm{GA})} = 4/3-4(n-1) \) of the GA approximation only for the lowest modes. The displacements fields corresponding to the five lowest eigenmodes in the limit of negligible adhesion are presented in Figure 2. The order \( n \) of the mode is equal to the number of its profile zeros including the one at \( y \rightarrow \infty \).

\[

\text{Figure 2: the displacements fields } u_y(y=0) \text{ corresponding to the five lowest SH eigenmodes in the absence of adhesion (} p_0 \neq 0 \text{). The order } n \text{ of the mode is equal to the number of its profile zeros including the one at } y \rightarrow \infty.\]

\[

\text{Figure 3: The wave depth profiles (displacement fields) for the } 1^{\text{st}} \text{ mode with the different normalized wave numbers } q \text{ (} q=0.3, 0.5, 1, 5, 10 \text{). The dependence on the wave number } q \text{ appears due to the surface loading (} p_0 \neq 0 \text{), in contrast to the case } p_0=0, \text{ when the wave forms depend on the combination } Cq^4.\]

With increasing pressure \( p_0 \neq 0 \) the surface \( \xi=\xi_n \) approaches the material’s surface because of \( q \sim \xi_n \). It follows that the description of the displacement field \( u_y \) includes both Kummer functions in Eq. (2) only when the phase velocity of SH acoustic modes satisfies the inequality \( c_{\text{sh}} > 2 \xi_n \). The displacement field for the slower modes is described in the entire half space \( 0 < \xi < \infty \) just by the second part of Eq. (2) and to determine \( d_n \) the Kummer function \( M \) should be omitted in Eq. (3), correspondingly. In the inset in Figure 2 the dependence of the eigenvalues on the dimensionless wave number \( q \) is presented. Accordingly, the increase of pressure contributes to the increase of the wave velocity and modifies the zero-pressure dispersion relation \( c_{\text{sh}} = c_{\text{cr}} q \). The wave depth profiles (see Figure 3 for the 1st mode) also start to be both pressure dependent and wave-number dependent even along the normalized depth coordinate (in Figure 3 \( q=k_y y + k_z p_z \) and \( \xi \sim y/(\rho y) \)). With increasing pressure these modes penetrate deeper into the medium. The self-similarity of the in-depth profiles is lost.
Localized modes polarized in the sagittal plane

In elastically homogeneous materials the longitudinal and shear waves interact only at the boundaries. In contrast, for the elastic P+SV modes propagating in the granular medium under gravity, due to the existing gradients of the elastic moduli, the longitudinal acoustic components are coupled to the shear ones in each point of the volume.

For an arbitrary power-law vertical distribution of the elastic moduli, the dispersion relations and the wave field distributions both for SH and P-SV modes can be obtained by expanding the displacement fields in the series of the Laguerre polynomials $L_n$ that provide a complete system of orthogonal functions for the half-space. In the case of adhesion neglected, the eigenvalues of the modes polarized in the sagittal plane are presented in Figure 4 for $\alpha=1/4$ and $\alpha=1/6$. The inset on the figure illustrates the dependence $d_\alpha=d_\alpha(\delta, \delta_c c_0^2/c_{30}^2)$ for the two lowest order modes.

To compare the theoretical predictions with the available measurement of the velocity $c_{\text{exp}}$ for the P+SV mode [11] we present the derived solution in the dimensional form

$$c_{\text{exp}} = \gamma_2 \left( \rho g d_\alpha / (p_{\text{exp}} k) \right)^{\alpha} \quad \text{and exclude the unknown parameter } \gamma_2$$

using experimental data on the pressure dependence of the bulk shear acoustic velocity $c_{\text{exp}} = \gamma_2 p_{\text{exp}}^{\alpha}$. Thus we obtain

$$c_{\text{exp}} = c_{\text{exp}} \left[ \rho g d_\alpha / (p_{\text{exp}} k) \right]^{\alpha} \quad \text{with } \rho = 1560 \text{ kg/m}^3, \quad c_{\text{exp}} = 800 \text{ m/s}, \quad c_{\text{exp}} = 1400 \text{ m/s}, \quad \delta = 3, \quad \alpha = 1/4 \text{ at } p_{\text{exp}} = 20 \text{ MPa}$$

from the data presented in [1,6], and for $k = 20 \text{ m}^{-1}$ (the wavelength $\lambda = 0.1 \text{ m}$) from the experimental range of [11] it has been found $c_{\text{exp}} = 45 \text{ m/s}$. Note that the dependence of the predicted magnitude of the velocity on $d_\alpha$ is very weak. With the calculated eigenvalue $d_1(\delta_c c_0^2/c_{30}^2) = 1$ for $\alpha = 1/4$ (see Figure 4), the theoretical prediction for the lowest mode $c_{\text{exp}} = 45 \text{ m/s}$ is very close to the experimental value $c_{\text{exp}}(\lambda = 0.1 \text{ m}) = 40 \text{ m/s}$.

Discussion

As it follows already from the dimensional analysis of both SH and P+SV modes in the vicinity of the free surface of granular media, their peculiar feature is the dependence of their velocity on the propagation constant $q$ (velocity dispersion) in the absence of the characteristic scale in the system. From this point of view, they differ importantly both from the Rayleigh acoustic wave on the surface of an elastically homogeneous material occupying a half space and from the SH and Lamb waveguide modes in plates. The former in the absence of a characteristic scale is non-dispersive, while the dispersion of the latter is entirely due to scale introduced by the plate thickness. In the case with adhesion ($p_\alpha > 0$), the physics of the P+SV modes propagating along the surface could be expected to differ from that of the SH modes due to possible localization of the P+SV wave near the surface, even in the absence of the vertical stratification. However, the fact that the Rayleigh-type mechanism of localization is not essential for the P+SV modes near the granular surface is implicitly confirmed by the fact that the SH modes are localized in the granular channel but not near the surface of elastically homogeneous media.

Conclusion

There exist several natural waveguides on the Earth, such as the deep sound channel in the ocean, channel for the radio-waves between the Earth’s surface and the ionosphere, channels for seismic waves propagating through the interior of the Earth, and others. In this article we propose a theory for another type of natural sound channels, guiding the surface waves propagating in a granular material with the gravity-caused variations of the elastic moduli.

References