Effective speed of shear waves in phononic crystals

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The effective shear-wave speed $c$ in 2D phononic crystals is considered. The two-sided explicit bounds converging to the exact value of $c$ from above and below are obtained via the plane-wave expansion and the monodromy-matrix methods. Convergence of the latter method is uniformly faster than of the former one. Comparative examples of using both methods of evaluating $c$ are presented.

1 Introduction

The paper analyses two methods of calculation of the effective (quasistatic) speed $c$ of shear waves in 2D phononic crystals: one is a common approach of plane wave expansion (PWE) [1], the other is based on the monodromy matrix (MM) [2, 3]. For each method, we define the upper and lower bound sequences which monotonically converge to the exact value of $c$. It is established that, for any fixed step, the pair of MM bounds lies in between the PWE bounds and thus provides a more accurate capture of the exact $c$. The proofs of the main theoretical results of the paper are omitted and will be reported elsewhere; instead we present a number of diagrams of the MM and PWE bounds of effective speed versus concentration of inclusions for several examples of two- and three-phase periodic lattices.

2 Background

We consider the wave equation

$$Cu \equiv -\nabla \cdot \mu \nabla u = \rho \omega^2 u. \quad (2.1)$$

Here $\mu$ and $\rho$ are real positive 1-periodic functions

$$\mu(x + e_i) = \mu(x), \quad \forall x \in \mathbb{R}^2, \quad i = 1, 2; \quad e_i = (\delta_{ij})_{j=1}^2 \quad (2.2)$$

with Kronecker symbol $\delta$. Using standard procedure of decomposition into the direct integral (see e.g. [5]) reveals that $C$ is unitarily equivalent to the operator $\int_{\mathbb{R}^2} \mathbb{C}(k) \, dk$ with $\mathbb{C}(k) = \mathbb{C}(k) = -i(k + \mathbf{i}) \cdot \mu(k + \mathbf{i})$ acting in $L^2([0, 1]^2)$ (the space of quadratic-summable functions on $[0, 1]^2$). It can be shown that $\mathbb{C}(k)$ has purely discrete spectrum $\omega_n^2(k) \leq \omega_n^2(k) \leq \ldots$, where $\omega_n(k)$ are called Floquet branches. Note that $\omega_1(0) = 0$ is an eigenvalue of $C(x)$ with multiplicity 1 and the corresponding eigenfunction is $v_1 \equiv \text{const}$. The effective speed is introduced as

$$c(k) = \lim_{k \to \infty} \frac{\omega_1(k)}{k}, \quad \text{where } k = k \mathbf{e}, \quad \|k\| = 1. \quad (2.3)$$

Using expansion

$$\mathbb{C}(k) = \mathbb{C}_0 + k \mathbb{C}_1 + k^2 \mathbb{C}_2, \quad \mathbb{C}_0 v = -\nabla \cdot \mu \nabla v, \quad \mathbb{C}_1 v = -i(k \cdot \mu \nabla v - \mu \nabla v), \quad \mathbb{C}_2 v = \mu v \quad (2.4)$$

and applying the perturbation theory to (2.1) defines $c(k)$ by the formula

$$c^2(k) = \frac{\langle \mu \rangle - (C_0^{-1} C_1 v_1, v_1)}{\langle \rho \rangle} = \frac{k \cdot \mu \text{eff} \hat{k}}{\langle \rho \rangle}, \quad (2.5)$$

where $\langle u, v \rangle = \langle u \overline{v} \rangle$ denotes the standard scalar product in $L^2([0, 1]^2)$ and $\langle \cdot \rangle = \int_{[0, 1]^2} \, dx$ is averaging. The real matrix $\mu \text{eff}$ is uniquely determined by (2.5). This formula requires calculation of the inverse of operator $\mathbb{C}(0)$, which in general has no exact representation except for some special cases (see an example in Appendix).

Hereafter we restrict consideration to the typical case of $\mu$ satisfying cubic symmetry $\mu(\sigma) = \mu(\cdot)$, where $\sigma$ is a matrix of rotation by $\frac{\pi}{2}$. In this case

$$\mu \text{eff} = \mu \text{eff} \mathbb{I} \quad (2.6)$$

($\mathbb{I}$ is $2 \times 2$ identity matrix) and hence effective speed $c^2 = \mu \text{eff} / \langle \rho \rangle$ does not depend on $k$.

Assumption of cubic symmetry allows us to use the following property (see [4]). Consider the problem

$$\tilde{C}u \equiv -\nabla \cdot \mu^{-1} \nabla u = \rho \omega^2 u. \quad (2.7)$$

Then the corresponding effective speed is $c^2 = \tilde{\mu} \text{eff} / \langle \rho \rangle$ with

$$\tilde{\mu} \text{eff} = \mu \text{eff}^{-1} \quad (2.8)$$

Note that formula (2.8) and thus the results of §3 can be modified for a general anisotropic case where $\mu \text{eff}$ is a matrix.

3 Two-sided estimates of $\mu \text{eff}$

3.1 PWE method

This method is based on using the formula (2.5) with $\mathbb{C}_0, \mathbb{C}_1$ restricted to the space of first $(2N + 1)^2$ simple harmonics $e^{2\pi i k \cdot x}$. For any function $r \in L^2([0, 1]^2)$ we denote its Fourier coefficients as $\tilde{r}$, i.e.

$$r(x) = \sum_{\mathbf{g} \in \mathbb{Z}^2} \tilde{r}(\mathbf{g}) e^{2\pi \mathbf{g} \cdot x}. \quad (3.9)$$

Introduce the finite matrix and vector

$$\mathbb{C}_{NN} \equiv (\langle \tilde{\mu}(g - g') \cdot g' \rangle)_{g, g' \in \epsilon_{SN}} \quad (3.10)$$

$$S_N = \{g = (g_1, g_2) \in \mathbb{Z}^2 \setminus \{0\} : |g_j| \leq N, \ j = 1, 2\} \quad (3.11)$$

Define

$$\mu_{NN} = \langle \mu \rangle - \mathbb{C}_{NN}^{-1} \mathbb{C}_{NN} \epsilon_{SN}, \quad (3.12)$$

see [1]. Doing the same with function $\tilde{r} \equiv \mu^{-1}$ we define $\tilde{\mu}_{NN}$. Now we formulate the first result.

Theorem 3.1. The sequence $\tilde{\mu}_{NN}^{-1}$ monotonically increases to $\mu \text{eff}$, the sequence $\mu_{NN}$ monotonically decreases to $\mu \text{eff}$, i.e.

$$\tilde{\mu}_{NN}^{-1} \nearrow \mu \text{eff}, \quad \mu_{NN} \searrow \mu \text{eff}, \quad N \to \infty. \quad (3.13)$$

For $N = 0$, (3.13) provides the Voigt-Reuss fork (see [4])

$$\langle \mu^{-1} \rangle \equiv \mu^{-1}, \quad \langle \mu^{-1} \rangle \equiv \mu \text{eff} \quad (3.14)$$

The PWE method in principle allows us to calculate $\mu \text{eff}$ with any desired accuracy, but in fact the PWE sequences $\mu_{NN}$ and $\tilde{\mu}_{NN}^{-1}$ converge slowly.
3.2 MM method

For any function \( r(x_1, x_2) \in L^2([0, 1]^2) \) let us denote its Fourier coefficients in \( x_2 \) as \( \hat{r}_n(x_1) \), i.e.
\[
r(x_1, x_2) = \sum_{n \in \mathbb{Z}} \hat{r}_n(x_1)e^{2\pi inx_2}.
\]

Introduce the matrices
\[
\tilde{r}_N \equiv \tilde{r}_N(x_1) = (\hat{r}_{m-n}(x_1))^N_{m=-N},
\]
\[
\partial_N = 2\pi \text{diag}(n)^N_{n=-N},
\]
where \( r \) is a function from \( L^2([0, 1]^2) \). Define \((2N + 1) \times (2N + 1)\) matrix
\[
Q_N = \left( \partial_N \tilde{m}_N \partial_N \right),
\]
where \( \tilde{m}_N \) is given by (3.16) applied to the function \( \tilde{m} \cdot 1 \) matrix. Denote
\[
\mu_N = e' \cdot (M_N - I)^{-1} e,
\]
\[
e = \left( e_0(N) \right), \quad e' = \left( 0 \right) \quad e_0(N) = (\delta n)^N_{n=-N}.
\]

Applying (3.17)-(3.19) to the function \( \tilde{e} \equiv \mu^{-1} \) yields the numbers \( \tilde{\mu}_N \). Now we formulate the main result.

**Theorem 3.2.** i) The sequence \( \tilde{\mu}_N^{-1} \) monotonically increases to \( \mu_{\text{eff}} \), the sequence \( \mu_N \) monotonically decreases to \( \mu_{\text{eff}} \), i.e.
\[
\tilde{\mu}_N^{-1} \uparrow \mu_{\text{eff}}, \quad \mu_N \downarrow \mu_{\text{eff}}, \quad N \to \infty.
\]

ii) Moreover,
\[
\tilde{\mu}_N^{-1} \leq \tilde{\mu}_N^{-1} \leq \mu_{\text{eff}} \leq \mu_N \leq \mu_{\text{NN}}, \quad \forall N,
\]
\( i.e. the bounds (3.20) yield a better approximation of \( \mu_{\text{eff}} \) than (3.13). \)

For \( N = 0 \) (3.20) gives us the known estimate (see [4])
\[
\langle (\mu^{-1})_2 \rangle_1 \leq \mu_{\text{eff}} \leq \langle (\mu^{-1})_2 \rangle_1^{-1}.
\]

Note that (3.19) admits a simpler form if \( \mu \) is even function. Denote the multiplicative integral over half of the period as
\[
M_{N, \frac{1}{2}} = \int_{0}^{\frac{1}{2}} (I + Q_N dx_1)
\]
and let \( m_N \) be the upper right \((2N + 1) \times (2N + 1)\) block of \( M_{N, \frac{1}{2}} \). Applying (3.17) and (3.23) to the function \( \tilde{\mu} \equiv \mu^{-1} \) defines \( \tilde{m}_N \).

**Theorem 3.3.** Suppose that \( \mu(x_1, x_2) = \mu(x_1, x_2) \) for all \( x_1, x_2 \). Then \( \mu_N, \tilde{\mu}_N \) which appear in (3.20) can also be defined by
\[
2\mu_N = e_0(N) \cdot m^{-1}_N e_0(N), \quad 2\tilde{\mu}_N = e_0(N) \cdot \tilde{m}_N^{-1} e_0(N),
\]
where \( e_0(N) \) are given by (3.19).

4 Examples

We present several examples of the PWE and MM bounds of effective speed evaluated for different \( N \) as functions of filling fraction in two- and three-phase lattices with high-contrast components. In the diagrams, the blue/dark blue curves are PWE upper and lower bounds \( \sqrt{\mu_{NN}/\rho} \) and \( \sqrt{\mu^{-1}_{NN}/\rho} \), respectively, and the red/brown curves are MM upper and lower bounds \( \sqrt{\mu_N/\rho} \) and \( \sqrt{\mu^{-1}_N/\rho} \), respectively.

It is observed that MM bounds provide a significantly sharper estimation of the exact effective speed. The fork of PWE bounds is relatively broader. However, for the two-phase lattices one of the PWE bounds is close to the exact effective speed, see Figs 1.b and 2.b. This is no longer so for three-phase lattices, see Figs 3 and 5.
Figure 2: PWE and MM bounds for Epoxy/Steel lattices of nested squares: a) $N = 0$, b) $N = 4$.

Figure 3: PWE and MM bounds for Steel/Epoxy/Silicium lattices of nested squares: a) $N = 0$, b) $N = 3$. 

$N = 4$
5 Conclusions

Let us recap the strong and weak points of MM and PWE methods:
1) MM is more accurate than PWE (see (3.21) and the figures).
2) Implementation of PWE is more straightforward than MM (see (3.10)-(3.12) and (3.17)-(3.18)).
3) MM requires less computation time per a step than PWE, since:
   MM needs to calculate an exponent of \((4N + 2) \times (4N + 2)\) matrix and to solve a system of \((4N + 2)\) linear equations,
   PWE needs to solve a system of \((2N + 1)^2\) linear equations.
6 Appendix

6.1 Example of a closed-form $\mu_{\text{eff}}$

Suppose that $\mu = \mu_1(x_1)\mu_2(x_2)$. Then $\mu_{\text{eff}}$ admits a closed-form solution

$$\mu_{\text{eff}} = \begin{pmatrix} \langle \mu_2 \rangle_2 (\mu_1^{-1})_1^{-1} & 0 \\ (\mu_1)_1 (\mu_2^{-1})_2^{-1} & 0 \end{pmatrix},$$

(6.25)

where $\langle \cdot \rangle_i = \int_0^1 \cdot \, dx_i$. In particular if $\mu$ depends on $x_1$ only, then $\langle \rho \rangle_2 = (\mu^{-1})^{-1} \kappa^2 + \langle \mu \rangle_2^2$.

6.2 Options for calculating the multiplicative integral

1. Consider the interval $[0, 1] = \cup^k_{n=1} \Delta_n$, $\Delta_n = [x_{n-1}, x_n]$, $0 = x_0 < x_1 < \ldots < x_k = 1$ and assume that the matrix-function $Q(x_1)$ does not depend on $x_1$ within $\Delta_n$. Then

$$M = \exp(|\Delta_k|Q(x_{k-1})) \ldots \exp(|\Delta_1|Q(x_0)).$$

(6.26)

A similar formula holds for $M_{N, \frac{1}{2}}$ (3.23) with the interval $[0, \frac{1}{2}]$ instead of $[0, 1]$.

2. Multiplicative integral (3.18) (or (3.23), with the interval $[0, \frac{1}{2}]$) can be calculated through Peano series

$$M = I + \int_0^1 Q(y_1) dy_1 + \int_0^1 \int_0^1 Q(y_1)Q(y_2) dy_1 dy_2 + \ldots,$$

(6.27)

which converges at the same rate as series for exponent of $Q$.

3. The definition of the multiplicative integral

$$M = \lim_{k \to \infty} \prod_{j=1}^{k} (I + (1/k)Q(j/k))$$

(6.28)

can also be applied for numerical calculation of its value.

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References


