Effective speed of shear waves in phononic crystals

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The effective shear-wave speed c in 2D phononic crystals is considered. The two-sided explicit bounds converging to the exact value of c from above and below are obtained via the plane-wave expansion and the monodromy-matrix methods. Convergence of the latter method is uniformly faster than of the former one. Comparative examples of using both methods of evaluating c are presented.

1 Introduction

The paper analyses two methods of calculation of the effective (quasistatic) speed c of shear waves in 2D phononic crystals: one is a common approach of plane wave expansion (PWE) [1], the other is based on the monodromy matrix (MM) [2, 3]. For each method, we define the upper and lower bound sequences which monotonically converge to the exact value of c. It is established that, for any fixed step, the pair of MM bounds lies in between the PWE bounds and thus provides a more accurate capture of the exact c. The proofs of the main theoretical results of the paper are omitted and will be reported elsewhere; instead we present a number of diagrams of the MM and PWE bounds of effective speed versus concentration of inclusions for several examples of two- and three-phase periodic lattices.

2 Background

We consider the wave equation

$$\mathcal{C}u \equiv -\nabla \cdot \mu \nabla u = \rho \omega^2 u. \tag{2.1}$$

Here μ and ρ are real positive $\mathbf{1}$-periodic functions

$$\mu, \rho(\mathbf{x} + \mathbf{e}_i) = \mu, \rho(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^2, \quad \forall i = 1, 2; \quad \mathbf{e}_i = (\delta_{ij})_{j=1}^2 \tag{2.2}$$

with Kronecker symbol δ. Using standard procedure of decomposition into the direct integral (see e.g. [5]) reveals that $\mathcal{C}$ is unitarily equivalent to the operator $\int^{\oplus}_{\mathbf{k}\in[-\pi,\pi]^2} \mathcal{C}(\mathbf{k})$ with $\mathcal{C}(\mathbf{k}) = -(\nabla + i\mathbf{k}) \cdot \mu(\nabla + i\mathbf{k})$ acting in $L^2([0,1]^2)$ (the space of quadratic-summable functions on $[0,1]^2$). It can be shown that $\mathcal{C}(\mathbf{k})$ has purely discrete spectrum $\omega_1^2(\mathbf{k}) \leqslant \omega_2^2(\mathbf{k}) \leqslant ...$, where $\omega_n(\mathbf{k})$ are called Floquet branches. Note that $\omega_1(\mathbf{0}) = 0$ is an eigenvalue of $\mathcal{C}(\mathbf{0})$ with multiplicity 1 and the corresponding eigenfunction is $v_1 \equiv \text{const}$. The effective speed is introduced as

$$c(\boldsymbol{\kappa}) = \lim_{k\to 0} \frac{\omega_1(\mathbf{k})}{k}, \quad \text{where} \quad \mathbf{k} = k\boldsymbol{\kappa}, \quad \|\boldsymbol{\kappa}\| = 1. \tag{2.3}$$

Using expansion

$$\mathcal{C}(\mathbf{k}) = \mathcal{C}_0 + k\mathcal{C}_1 + k^2\mathcal{C}_2, \quad \mathcal{C}_0 v = -\nabla \cdot \mu \nabla v, \quad \mathcal{C}_1 v = -i\boldsymbol{\kappa} \cdot \mu \nabla v - i\nabla \cdot \mu \boldsymbol{\kappa} v, \quad \mathcal{C}_2 v = \mu v \tag{2.4}$$

and applying the perturbation theory to (2.1) defines $c(\boldsymbol{\kappa})$ by the formula

$$c^2(\boldsymbol{\kappa}) = \frac{\langle\mu\rangle - (\mathcal{C}_1\mathcal{C}_0^{-1}\mathcal{C}_1 v_1, v_1)}{\langle\rho\rangle} \equiv \frac{\boldsymbol{\kappa} \cdot \boldsymbol{\mu}_{\text{eff}} \boldsymbol{\kappa}}{\langle\rho\rangle}, \tag{2.5}$$

where $(u, v) = \langle u\overline{v}\rangle$ denotes the standard scalar product in $L^2([0,1]^2)$ and $\langle\cdot\rangle = \int_{[0,1]^2} \cdot d\mathbf{x}$ is averaging. The real matrix $\boldsymbol{\mu}_{\text{eff}}$ is uniquely determined by (2.5). This formula requires calculation of the inverse of operator $\mathcal{C}(\mathbf{0})$, which in general has no exact representation except for some special cases (see an example in Appendix).

Hereafter we restrict consideration to the typical case of μ satisfying cubic symmetry $\mu(\boldsymbol{\sigma}\cdot) = \mu(\cdot)$, where $\boldsymbol{\sigma}$ is a matrix of rotation by $\frac{\pi}{2}$. In this case

$$\boldsymbol{\mu}_{\text{eff}} = \mu_{\text{eff}} \mathbf{I} \tag{2.6}$$

($\mathbf{I}$ is 2×2 identity matrix) and hence effective speed $c^2 = \mu_{\text{eff}}/\langle\rho\rangle$ does not depend on $\boldsymbol{\kappa}$.

Assumption of cubic symmetry allows us to use the following property (see [4]). Consider the problem

$$\widetilde{\mathcal{C}}u \equiv -\nabla \cdot \mu^{-1} \nabla u = \rho\omega^2 u. \tag{2.7}$$

Then the corresponding effective speed is $\widetilde{c}^2 = \widetilde{\mu}_{\text{eff}}/\langle\rho\rangle$ with

$$\widetilde{\mu}_{\text{eff}} = \mu_{\text{eff}}^{-1}. \tag{2.8}$$

Note that formula (2.8) and thus the results of §3 can be modified for a general anisotropic case where $\boldsymbol{\mu}_{\text{eff}}$ is a matrix.

3 Two-sided estimates of μ_{eff}

3.1 PWE method

This method is based on using the formula (2.5) with $\mathcal{C}_0$, $\mathcal{C}_1$ restricted to the space of first $(2N+1)^2$ simple harmonics $e^{2\pi i \mathbf{g}\cdot\mathbf{x}}$. For any function $r \in L^2([0,1]^2)$ we denote its Fourier coefficients as $\widehat{r}$, i.e.

$$r(\mathbf{x}) = \sum_{\mathbf{g}\in\mathbb{Z}^2} \widehat{r}(\mathbf{g}) e^{2\pi i \mathbf{g}\cdot\mathbf{x}}. \tag{3.9}$$

Introduce the finite matrix and vector

$$\mathcal{C}_{NN} \equiv (\widehat{\mu}(\mathbf{g} - \mathbf{g}')\mathbf{g} \cdot \mathbf{g}')_{\mathbf{g},\mathbf{g}'\in S_N}, \quad \mathbf{f}_{NN} = (\widehat{\mu}(\mathbf{g}) g_1)_{\mathbf{g}\in S_N}, \tag{3.10}$$

where

$$S_N = \{\mathbf{g} = (g_1, g_2) \in \mathbb{Z}^2 \setminus \{\mathbf{0}\} : \ |g_j| \leqslant N, \ j = 1, 2\}. \tag{3.11}$$

Define

$$\mu_{NN} = \langle\mu\rangle - \mathbf{f}_{NN} \cdot \mathcal{C}_{NN}^{-1} \mathbf{f}_{NN}, \tag{3.12}$$

see [1]. Doing the same with function $\widetilde{\mu} \equiv \mu^{-1}$ we define $\widetilde{\mu}_{NN}$. Now we formulate the first result.

Theorem 3.1. *The sequence $\widetilde{\mu}_{NN}^{-1}$ monotonically increases to μ_{eff}, the sequence μ_{NN} monotonically decreases to μ_{eff}, i.e.*

$$\widetilde{\mu}_{NN}^{-1} \nearrow \mu_{\text{eff}}, \quad \mu_{NN} \searrow \mu_{\text{eff}}, \quad N \to \infty. \tag{3.13}$$

For $N = 0$, (3.13) provides the Voigt-Reuss fork (see [4])

$$\langle\mu^{-1}\rangle^{-1} \leqslant \mu_{\text{eff}} \leqslant \langle\mu\rangle. \tag{3.14}$$

The PWE method in principle allows us to calculate μ_{eff} with any desired accuracy, but in fact the PWE sequences μ_{NN} and $\widetilde{\mu}_{NN}^{-1}$ converge slowly.

3.2 MM method

For any function $r(x_1, x_2) \in L^2([0,1]^2)$ let us denote its Fourier coefficients in x_2 as $\widehat{r}_n(x_1)$, i.e.

$$r(x_1, x_2) = \sum_{n\in\mathbb{Z}} \widehat{r}_n(x_1) e^{2\pi i n x_2}. \qquad (3.15)$$

Introduce the matrices

$$\widehat{\mathbf{r}}_N \equiv \widehat{\mathbf{r}}_N(x_1) = (\widehat{r}_{n-m}(x_1))_{n,m=-N}^{N},$$
$$\boldsymbol{\partial}_N = 2\pi\,\mathrm{diag}(n)_{-N}^{N}, \qquad (3.16)$$

where r is a function from $L^2([0,1]^2)$. Define $(2N+1)\times(2N+1)$ matrix

$$\mathbf{Q}_N = \begin{pmatrix} \mathbf{0} & \widehat{\boldsymbol{\mu}}_N^{-1} \\ \boldsymbol{\partial}_N\widehat{\boldsymbol{\mu}}_N\boldsymbol{\partial}_N & \mathbf{0} \end{pmatrix}, \qquad (3.17)$$

where $\widehat{\boldsymbol{\mu}}_N$ is given by (3.16) applied to the function μ. Introduce the multiplicative integral [6] (see Appendix)

$$\mathbf{M}_N = \overbrace{\int_0^1} (\mathbf{I} + \mathbf{Q}_N dx_1), \qquad (3.18)$$

where $\mathbf{I}$ is $(4N+2)\times(4N+2)$ identity matrix. Denote

$$\mu_N = \mathbf{e}' \cdot (\mathbf{M}_N - \mathbf{I})^{-1}\mathbf{e},$$
$$\mathbf{e} = \begin{pmatrix} \mathbf{e}_{(N)} \\ \mathbf{0} \end{pmatrix}, \quad \mathbf{e}' = \begin{pmatrix} \mathbf{0} \\ \mathbf{e}_{(N)} \end{pmatrix}, \quad \mathbf{e}_{(N)} = (\delta_{0n})_{-N}^{N}. \qquad (3.19)$$

Applying (3.17)-(3.19) to the function $\widetilde{\mu} \equiv \mu^{-1}$ yields the numbers $\widetilde{\mu}_N$. Now we formulate the main result

Theorem 3.2. *i) The sequence $\widetilde{\mu}_N^{-1}$ monotonically increases to μ_{eff}, the sequence μ_N monotonically decreases to μ_{eff}, i.e.*

$$\widetilde{\mu}_N^{-1} \nearrow \mu_{\text{eff}}, \quad \mu_N \searrow \mu_{\text{eff}}, \quad N \to \infty. \qquad (3.20)$$

ii) Moreover,

$$\widetilde{\mu}_{NN}^{-1} \leqslant \widetilde{\mu}_N^{-1} \leqslant \mu_{\text{eff}} \leqslant \mu_N \leqslant \mu_{NN}, \quad \forall N, \qquad (3.21)$$

i.e. the bounds (3.20) yield a better approximation of μ_{eff} than (3.13).

For $N = 0$ (3.20) gives us the known estimate (see [4])

$$\langle\langle\mu^{-1}\rangle_2^{-1}\rangle_1 \leqslant \mu_{\text{eff}} \leqslant \langle\langle\mu\rangle_2^{-1}\rangle_1^{-1}. \qquad (3.22)$$

Note that (3.19) admits a simpler form if μ is even function. Denote the multiplicative integral over half of the period as

$$\mathbf{M}_{N,\frac{1}{2}} = \overbrace{\int_0^{\frac{1}{2}}} (\mathbf{I} + \mathbf{Q}_N dx_1) \qquad (3.23)$$

and let $\mathbf{m}_N$ be the upper right $(2N+1)\times(2N+1)$ block of $\mathbf{M}_{N,\frac{1}{2}}$. Applying (3.17) and (3.23) to the function $\widetilde{\mu} \equiv \mu^{-1}$ defines $\widetilde{\mathbf{m}}_N$.

Theorem 3.3. *Suppose that $\mu(-x_1, x_2) = \mu(x_1, x_2)$ for all x_1, x_2. Then μ_N, $\widetilde{\mu}_N$ which appear in (3.20) can also be defined by*

$$2\mu_N = \mathbf{e}_{(N)} \cdot \mathbf{m}_N^{-1}\mathbf{e}_{(N)}, \quad 2\widetilde{\mu}_N = \mathbf{e}_{(N)} \cdot \widetilde{\mathbf{m}}_N^{-1}\mathbf{e}_{(N)}, \quad (3.24)$$

where $\mathbf{e}_{(N)}$ are given by (3.19).

4 Examples

We present several examples of the PWE and MM bounds of effective speed evaluated for different N as functions of filling fraction in two- and three-phase lattices with high-contrast components. In the diagrams, the blue/dark blue curves are PWE upper and lower bounds $\sqrt{\mu_{NN}/\langle\rho\rangle}$ and $\sqrt{\widetilde{\mu}_{NN}^{-1}/\langle\rho\rangle}$, respectively, and the red/brown curves are MM upper and lower bounds $\sqrt{\mu_N/\langle\rho\rangle}$ and $\sqrt{\widetilde{\mu}_N^{-1}/\langle\rho\rangle}$, respectively.

It is observed that MM bounds provide a significantly sharper estimation of the exact effective speed. The fork of PWE bounds is relatively broader. However, for the two-phase lattices one of the PWE bounds is close to the exact effective speed, see Figs 1.b and 2.b. This is no longer so for three-phase lattices, see Figs 3 and 5.

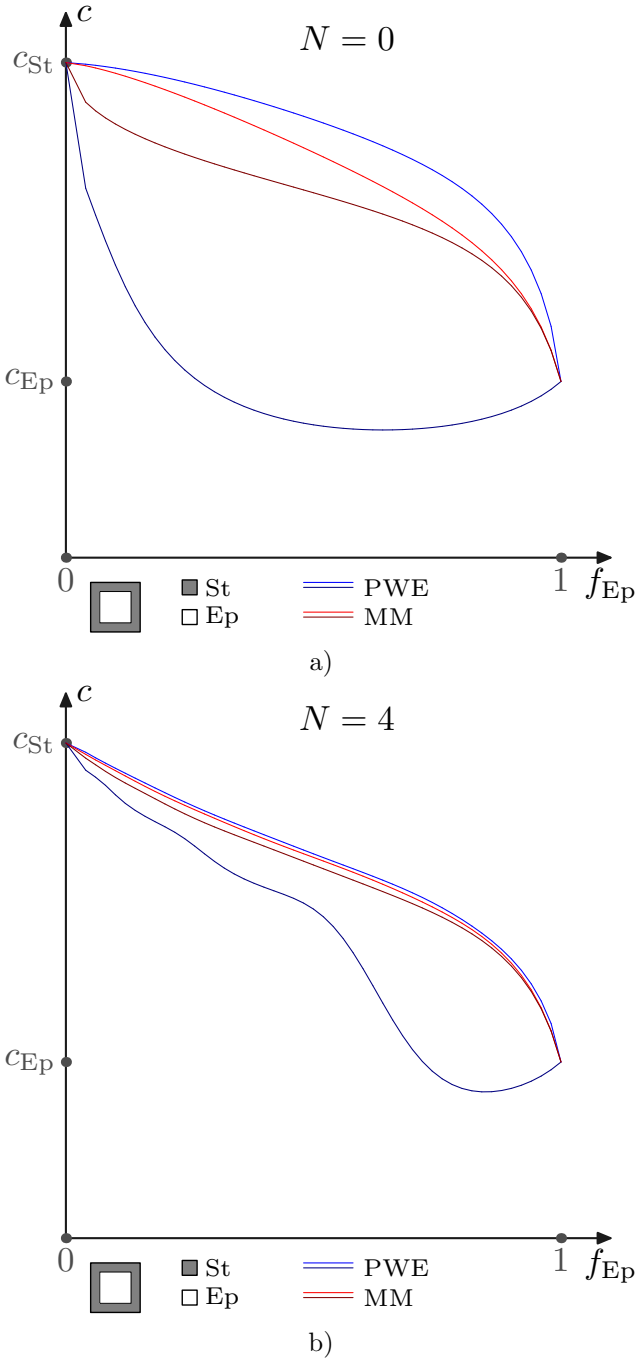


Figure 1: PWE and MM bounds for Steel/Epoxy lattice of nested squares: a) $N = 0$, b) $N = 4$.

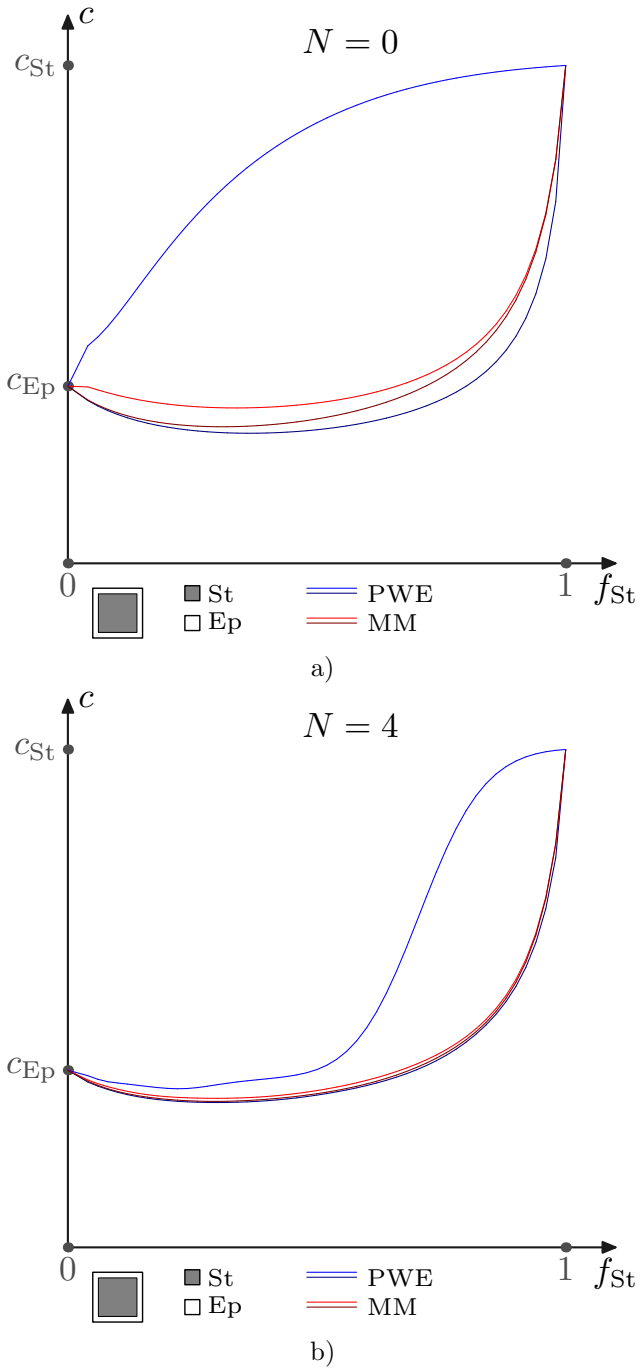


Figure 2: PWE and MM bounds for Epoxy/Steel lattices of nested squares: a) $N = 0$, b) $N = 4$.

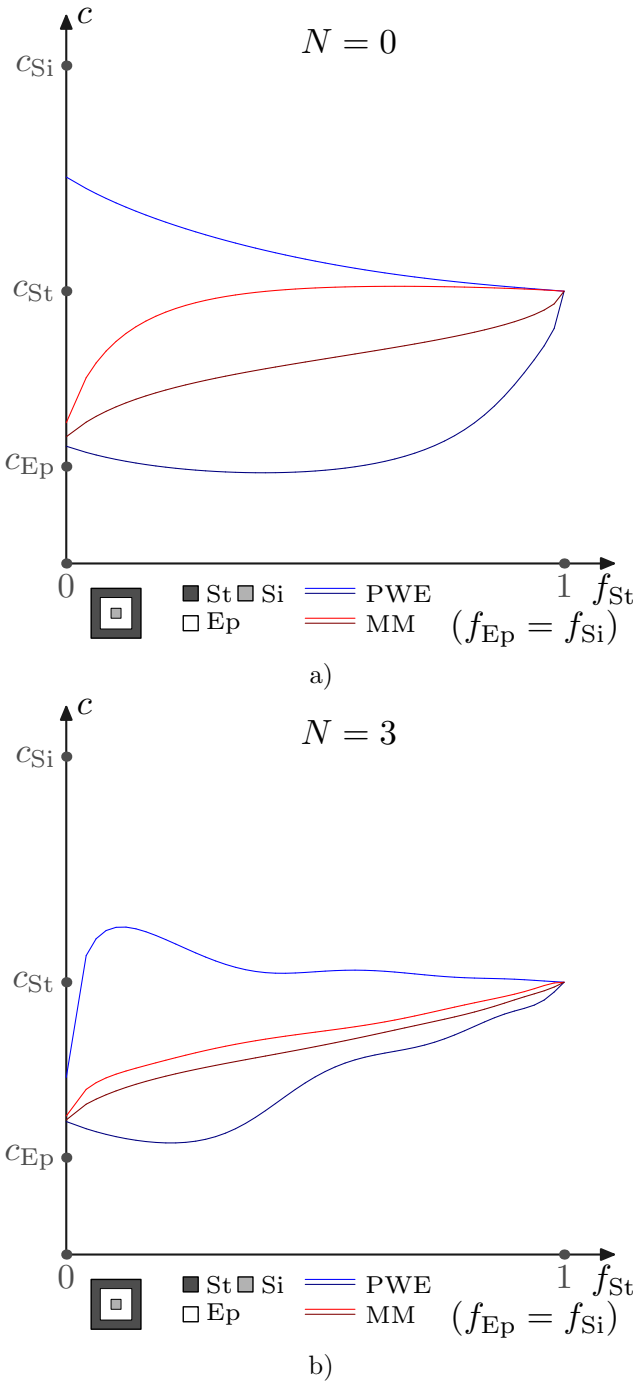
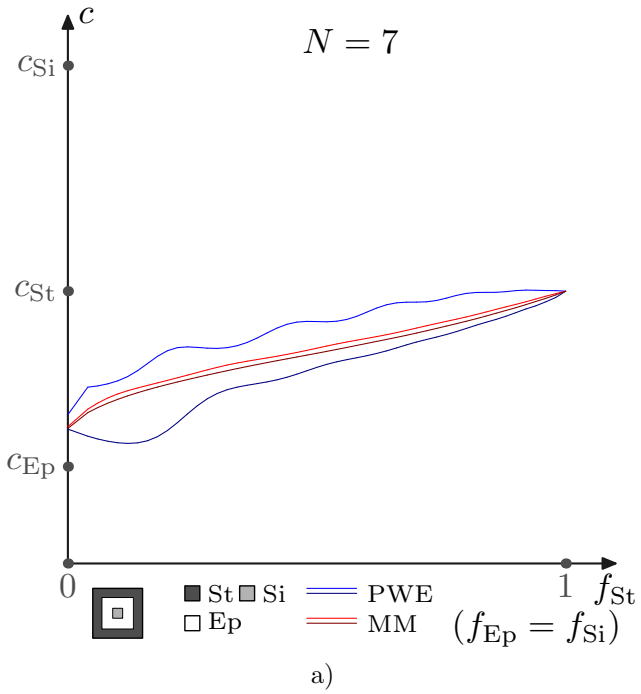
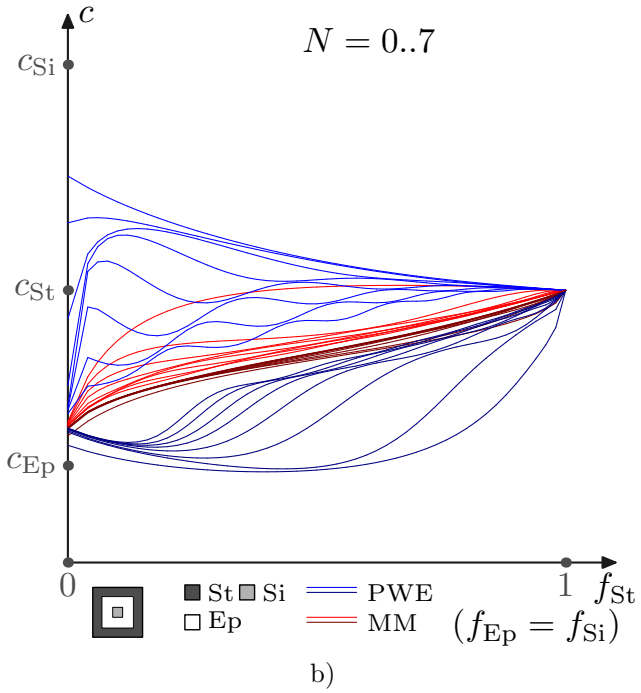


Figure 3: PWE and MM bounds for Steel/Epoxy/Silicium lattices of nested squares: a) $N = 0$, b) $N = 3$.

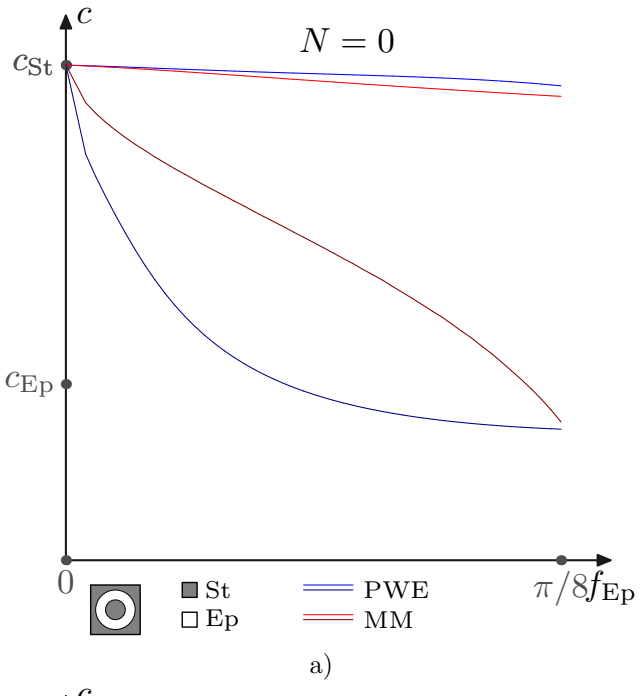


a)

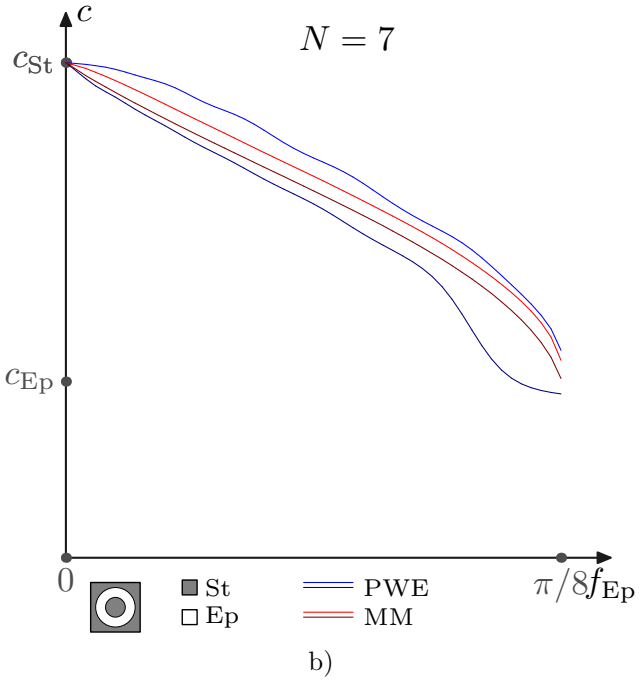


b)

Figure 4: PWE and MM bounds for Steel/Epoxy/Silicium lattices of nested squares: a) $N = 7$, b) $N = 0..7$.



a)



b)

Figure 5: PWE and MM bounds for Steel/Epoxy lattice of nested circles: a) $N = 0$, b) $N = 7$.

5 Conclusions

Let us recap the strong and weak points of MM and PWE methods:

1) MM is more accurate than PWE (see (3.21) and the figures).

2) Implementation of PWE is more straightforward than MM (see (3.10)-(3.12) and (3.17)-(3.18)).

3) MM requires less computation time per a step than PWE, since:

MM needs to calculate an exponent of $(4N + 2) \times (4N + 2)$ matrix and to solve a system of $(4N + 2)$ linear equations,

PWE needs to solve a system of $(2N + 1)^2$ linear equations.

6 Appendix

6.1 Example of a closed-form $\boldsymbol{\mu}_{\text{eff}}$

Suppose that $\mu = \mu_1(x_1)\mu_2(x_2)$. Then $\boldsymbol{\mu}_{\text{eff}}$ admits a closed-form solution

$$\boldsymbol{\mu}_{\text{eff}} = \begin{pmatrix} \langle\mu_2\rangle_2\langle\mu_1^{-1}\rangle_1^{-1} & 0 \\ 0 & \langle\mu_1\rangle_1\langle\mu_2^{-1}\rangle_2^{-1} \end{pmatrix}, \qquad (6.25)$$

where $\langle\cdot\rangle_i = \int_0^1 \cdot dx_i$. In particular if μ depends on x_1 only, then $\langle\rho\rangle c^2 = \langle\mu^{-1}\rangle^{-1}{\kappa_1}^2 + \langle\mu\rangle\kappa_2^2$.

6.2 Options for calculating the multiplicative integral

1. Consider the interval $[0,1] = \cup_1^k \Delta_n$, $\Delta_n = [x_{n-1}, x_n)$, $0 = x_0 < x_1 < ... < x_k = 1$ and assume that the matrix-function $\mathbf{Q}(x_1)$ does not depend on x_1 within Δ_n. Then

$$\mathbf{M} = \exp(|\Delta_k|\mathbf{Q}(x_{k-1}))... \exp(|\Delta_1|\mathbf{Q}(x_0)). \qquad (6.26)$$

A similar formula holds for $\mathbf{M}_{N,\frac{1}{2}}$ (3.23) with the interval $[0, \frac{1}{2}]$ instead of $[0,1]$.

2. Multiplicative integral (3.18) (or (3.23), with the interval $[0, \frac{1}{2}]$) can be calculated through Peano series

$$\mathbf{M} = \mathbf{I} + \int_0^1 \mathbf{Q}(y_1)dy_1 + \int_0^1 \int_0^{y_1} \mathbf{Q}(y_1)\mathbf{Q}(y_2)dy_1dy_2 + ..., \qquad (6.27)$$

which converges at the same rate as series for exponent of $\mathbf{Q}$.

3. The definition of the multiplicative integral

$$\mathbf{M} = \lim_{k\to\infty} \prod_{j=1}^{k} (\mathbf{I} + (1/k)\mathbf{Q}(j/k)) \qquad (6.28)$$

can also be applied for numerical calculation of its value.

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