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Proposition of a Green-Volterra formalism to solve dynamics of a nonlinear string

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The convolution kernels of a Volterra series give a generalization of impulse responses to the case of weakly nonlinear input-to-output systems. In acoustics, this formalism has yet been used to solve, e.g. the problem of a nonlinear string excited by a force $f(t)$ (considered as the input), spatially distributed by a time invariant function. In this paper, we propose a similar generalization for the case of Green functions in order to tackle inputs that depend on both the space and the time variables. The method to derive the "Green-Volterra" kernels is described and its application to the string is presented.

1 Introduction

The Volterra series can be used to compute dynamics of weakly nonlinear problems. A previous work [1] has presented the computation of Volterra kernels for a string model with excitation force defined as $\phi(x)f(t)$. This hypothesis is very restrictive for sound synthesis purposes since the excitation force is not spatially constant when it comes to a musician. A bow motion for instance, can be decomposed in several forces on the string according to several directions. These forces are dependent of how the musician wants to play. That is the reason why we need a method able to manage a much more general excitation force for the system.

Green's function is a good formalism to solve this kind of issues. Its main drawback is that it is an application of the superposition principle, and can only be used for linear problems. This paper introduces an extension of the Volterra series with Green's function formalism. The main purpose is to use Volterra series without the "spatially constant" hypothesis on the excitation force.

The first part will introduce both methods: Green's function for a linear problem, and Volterra series for the Kirchhoff string model. The second part will describe the theory of Green-Volterra kernels. Finally, an application to the same string model and a structure of simulation will be presented.

2 Problem statement

2.1 Model under consideration

Consider the dimensionless nonlinear Kirchhoff model of the transverse vibrations of a damped string [2] excited by a force f distributed on $\Omega =]0, 1[$ given by, for all $(x, t) \in \Omega \times \mathbb{R}_+$,

$$\partial_t^2 u(x, t) + 2\alpha \partial_t u(x, t) - \left(1 + \epsilon \left[\int_0^1 (\partial_x u(x, t))^2 dx \right]\right) \partial_x^2 u(x, t) = f(x, t), \quad (1)$$

with Dirichlet boundary conditions and zero initial conditions

$$\forall t > 0, \quad u(0, t) = 0 \text{ and } u(1, t) = 0, \quad (2)$$

$$\forall x \in \Omega, \quad u(x, 0) = 0 \text{ and } \partial_t u(x, 0) = 0. \quad (3)$$

$\epsilon > 0$ is the coefficient of the nonlinearity, which takes into account the variation of tension. $\alpha > 0$ accounts for the fluid damping.

2.2 Linearized problem: Green's function solution

The linearized version of (1-3) obtained for $\epsilon = 0$ is a standard well-posed problem that has been extensively

studied. Using the Green's function formalism, the solution of this causal problem can be expressed as

$$u(x, t) = \int_{\Omega \times \mathbb{R}_+} g(x, \tau, \xi) f(\xi, t - \tau) d\xi d\tau \quad (4)$$

where the Green's kernel g is the solution of (1-3) with $\epsilon = 0$, for $f(x, t) = \delta(x - \xi)\delta(t)$. In the Laplace domain (with respect to the time variable), the linear problem is $\forall (x, s) \in \Omega \times \mathbb{C}_{-2\alpha}^+$

$$\Gamma^2(s)U(x, s) - \partial_x^2 U(x, s) = F(x, s), \quad (5)$$

with

$$\Gamma(s) = \sqrt{s^2 + 2\alpha s}, \quad (6)$$

and the following boundary conditions

$$U(x = 0, s) = U(x = 1, s) = 0 \quad (7)$$

where, for all $x \in \mathbb{R}$, $\mathbb{C}_\sigma^+ = \{s \in \mathbb{C} \mid \Re(s) > \sigma\}$ and $z \mapsto \sqrt{z}$ denotes the analytic continuation over $\mathbb{C}_0^+$ of the positive square root on $\mathbb{R}_+$.

This linear boundary problem solves into for all $(x, s) \in \Omega \times \mathbb{C}_{-2\alpha}^+$ and $\xi \in \Omega$,

$$U(x, s) = \int_{\Omega} G(x, s, \xi) F(\xi, s) d\xi, \quad (8)$$

$$G(x, s, \xi) = \frac{\cosh((1 + x + \xi)\Gamma(s)) - \cosh((1 - |x - \xi|)\Gamma(s))}{2\Gamma(s) \sinh(\Gamma(s))} \quad (9)$$

The solution can also be decomposed on the eigenfunctions $\{e_k\}_{k \in \mathbb{N}^*}$ with $e_k(x) = \sqrt{2} \sin(k\pi x)$, which define an orthonormal (Hilbert) basis of $L^2(\Omega)$. If $\alpha < \pi$, the characteristic equation of the k -th mode is $\Gamma(s)^2 + k^2\pi^2 = 0$. It admits a pair of complex conjugate roots $(\lambda_k, \bar{\lambda}_k)$ with $\lambda_k = -\alpha + i\sqrt{\omega_k}$ and $\omega_k = \sqrt{k^2\pi^2 - \alpha^2} > 0$. The modal decomposition yields

$$u(x, t) = \sum_{k=1}^{+\infty} [g_k \star f_k](t) e_k(x), \quad (10)$$

$$U(x, s) = \sum_{k=1}^{+\infty} G_k(s) F_k(s) e_k(x), \quad (11)$$

with $g_k(t) = 1_{t>0}(t)e^{-\alpha t}$, $G_k(s) = \frac{1}{s^2 + 2\alpha s + k^2\pi^2}$, $f_k(t) = f_k(t) = \langle f(t, \cdot), e_k(\cdot) \rangle_{L^2(\Omega)}$ and $F_k(s) = \langle F(s, \cdot), e_k(\cdot) \rangle_{L^2(\Omega)}$ denote the projections of f and F on eigenfunction e_k .

For sound synthesis purposes, real-time simulations are often based on a truncated version of this modal decomposition. The block-diagram corresponding to (10-11) truncated at order K is displayed in figure 1. For simulations, digital versions of the second order autoregressive filters associated with the transfer function $G_k(s)$ are implemented.

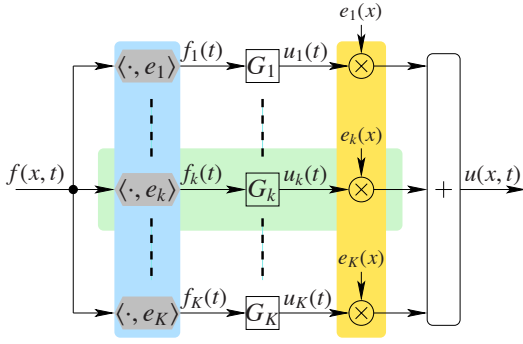


Figure 1: Block-diagram of the simulation of the linearized version of (1-3) with K modes.

2.3 Nonlinear problem for a class of excitations: Volterra series solution

The problem (1-3) has been solved in [1] using Volterra series expansions, in the case where $f(x, t) = \phi(x)f(t)$. The main steps of the method and the results are recalled below.

2.3.1 Volterra kernels

The definitions and results used in this paper are gathered from [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. The system defined by Eq. (1) is space dependent, therefore the Volterra series has space parameterized kernels $h_n^{(x)}$. The output u of the system, represented by stationary Volterra kernels $h_n^{(x)}$ is defined by:

$$u(x, t) = \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} h_n^{(x)}(\tau_1, \dots, \tau_n) u(t - \tau_1) \dots u(t - \tau_n) d\tau_1 \dots d\tau_n \quad (12)$$

where $(\tau_1, \dots, \tau_n) \in \mathbb{R}^n$ are the convolution variables for a n -order output and u the input of the system. If $\mathcal{T}$ is a Laplace or a Fourier transform, we will write $H_n^{(x)} = \mathcal{T}(h_n^{(x)})$.

The use of stationary Volterra kernels implies to write the excitation force

$$f(x, t) = \phi(x)g(t) \quad (13)$$

which gives the following results.

2.3.2 Equations of Volterra kernels

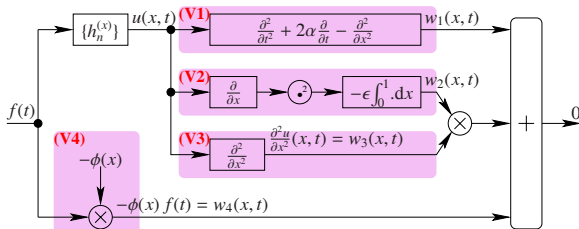


Figure 2: Equivalent block-diagrams representing Eq. (1).

The Volterra kernels will be determined using the canceling system (cf. figure 2) which represents Eq. (1) and the interconnection laws. For more detail, the reader can refer to [1], model (M1).

$$(\Gamma(s_1 + \dots + s_n))^2 H_n^{(x)}(s_{1:n}) - \partial_x^2 H_n^{(x)}(s_{1:n}) = E_n^{(x)}(s_{1:n}) \quad (14)$$

where $s_{1:n} = (s_1, \dots, s_n)$, with the following boundary conditions

$$\forall (s_{1:n}) \in \{0; 1\}(\mathbb{C}_0^+)^n, \quad H_n^{(x=0)}(s_{1:n}) = H_n^{(x=1)}(s_{1:n}) = 0 \quad (15)$$

and

$$E_1^{(x)}(s_1) = \phi(x) \quad (16)$$

$$E_n^{(x)}(s_{1:n}) = \epsilon \sum_{p+q+r=n} \int_0^1 [\partial_x H_p^{(x)}(s_{1:p}) \partial_x H_q^{(x)}(s_{p+1:p+q})] dx \quad (17)$$

Eqs. (14) to (17) are similar to system defined in (5) to (7) where $U(x, s) \mapsto H_n^{(x)}(s_{1:n})$, $F(x, s) \mapsto E_n^{(x)}(s_{1:n})$ and $\Gamma(s) \mapsto \Gamma(s_1 + \dots + s_n)$. It is the same boundary problem (with respect to x), as a consequence (similarly to Eq. (8)) $\forall (x, s_{1:n}) \in \Omega \times (\mathbb{C}_0^+)^n$:

$$H_n^{(x)}(s_{1:n}) = \int_{\Omega} G(x, s_1 + \dots + s_n, \xi) E_n^{(x)}(s_{1:n}) d\xi \quad (18)$$

Remark 1 For $n = 1$, the Volterra kernel H_1 is a convolution between the Green function G (cf. Eq. (9)) and $E_1 = \phi$ the spatial force distribution. Therefore H_1 gives the solution of the linearized problem ($\epsilon = 0$) since it is the Green function of linearized version of Eq. (1).

2.3.3 Modal projection

For all $n \in \mathbb{N}^*$, Eqs. (14) to (17) is similar to the problem defined in Eqs. (5) to (7), it can be therefore decomposed on a modal basis with the following definition valid in the set $L^2(\Omega)$

$$H_n^{(x)}(s_{1:n}) = \sum_{k \in \mathbb{N}^*} H_n^{[k]}(s_{1:n}) e_k(x) \quad (19)$$

with $e_k = \sqrt{2} \sin(k\pi x)$.

The decomposition of Eqs. (14- 17) on the eigenfunctions $\{e_k\}_{k \in \mathbb{N}^*}$ with $e_k = \sqrt{2} \sin(k\pi x)$ defines the modal projection of the Volterra kernels

$$H_n^{[k]}(s_{1:n}) = G^{[k]}(s_1 + \dots + s_n) E_n^{[k]}(s_{1:n}). \quad (20)$$

Eq. (20) is identical to Eq. (11) where $G_k \mapsto G^{[k]}$, $e_k \mapsto e_k$ and $F_k \mapsto E_n^{[k]}$ which leads to

$$E_1^{[k]}(s_1) = \langle \phi, e_k \rangle = \phi_k \quad (21)$$

$$E_n^{[k]}(s_{1:n}) = -\epsilon k^2 \pi^4 \sum_{p+q+r=n} \left[\sum_{l \in \mathbb{N}^*} H_p^{[l]}(s_{1:p}) H_q^{[l]}(s_{p+1:p+q}) \right] H_r^{[k]}(s_{p+q+1:r}) \quad (22)$$

As detailed in [1], this system can be rewritten using trees. The interested reader can refer to the mentioned article.

2.3.4 Numerical simulation in the time domain

Using Eq. (20), kernels can be identified as structures composed of linear systems, sums and products (cf. [1] for details).

Figure 3 shows the restriction on excitation forces applied in previous works. This force is defined as a product between a time and a space function: $f(x, t) = \phi(x)f(t)$. This imposes to know $\phi(x)$.

2.4 Nonlinear problem under consideration: what formalism for a general solution?

The generalization of an excitation force $f(x, t)$, is a necessity to deal with realistic physical problems. That is the point of the next section that will define the Green-Volterra kernels. Then, the method will be applied at the same string model introduced in this section.

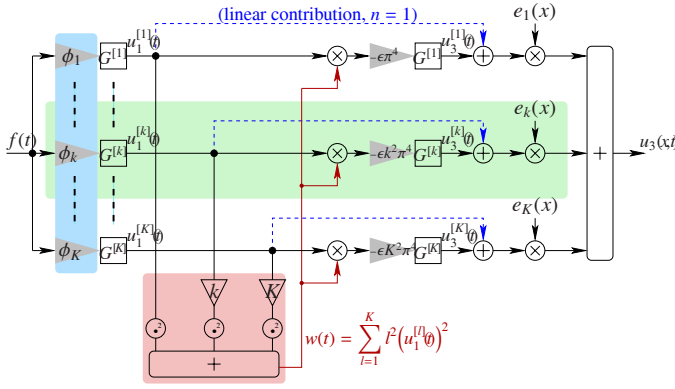


Figure 3: Block-diagram of an $o(\epsilon)$ -simulation of Eq. (1) with K modes: the dashed arrows isolate the linear dynamics ($n=1$) of each mode; the shaded central part isolates the $o(\epsilon)$ -dynamics of a mode k ; the shaded bottom part isolates the dynamics of the integral term in (M1) (gains 1, k and K before the square operators are due to ∂_x); the shaded left part corresponds to simple gains, controlled by the spatial distribution of the excitation force.

3 Introducing Green-Volterra series

In this section, we propose to generalize the Volterra series formalism presented in [1], in order to tackle the nonlinear problem (1-3).

3.1 Definitions

Definition 1 (Green-Volterra series) A system with input $f : (x, t) \in \Omega \times \mathbb{R} \mapsto f(x, t)$ and output $u : (x, t) \in \Omega \times \mathbb{R} \mapsto u(x, t)$ is described by a Green-Volterra series of kernels $\{r_n\}_{n \in \mathbb{N}^*}$, if the output is given by

$$u(x, t) = \sum_{n=1}^{+\infty} \int_{(\Omega \times \mathbb{R})^n} r_n(x, t, \xi_{1:n}, \tau_{1:n}) \times f(\xi_1, \tau_1) \dots f(\xi_n, \tau_n) d\xi_{1:n} d\tau_{1:n} \quad (23)$$

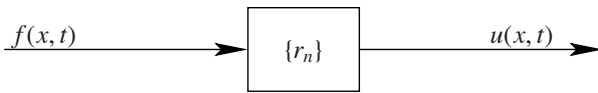


Figure 4: System represented by its Green-Volterra kernels

For causal time-invariant systems, definition 1 can be simplified and transfer kernels can be introduced in the Laplace domain, as for standard Volterra series. This leads to the following definitions.

Definition 2 (Time-invariant system and transfer kernels) If the system defined by (23) is invariant under time translations, then for all $n \in \mathbb{N}^*$ and $(x, t, \xi_1, \tau_1, \dots, \xi_n, \tau_n) \in (\Omega \times \mathbb{R})^{1+n}$,

$$r_n(x, t, \xi_1, \tau_1, \dots, \xi_n, \tau_n) = r_n(x, 0, \xi_1, \tau_1 - t, \dots, \xi_n, \tau_n - t), \quad (24)$$

and (23) takes the following form

$$u(x, t) = \sum_{n=1}^{+\infty} \int_{(\Omega \times \mathbb{R})^n} r_n(x, \xi_{1:n}, \tau_{1:n}) \times f(\xi_1, t - \tau_1) \dots f(\xi_n, t - \tau_n) d\xi_{1:n} d\tau_{1:n} \quad (25)$$

where $r_n(x, \xi_1, \tau_1, \dots, \xi_n, \tau_n) = r_n(x, 0, \xi_1, \tau_1, \dots, \xi_n, \tau_n)$. In this case, for all $(\xi_1, \dots, \xi_n)$, the Green-Volterra transfer kernel R_n is defined by the multi-variate Laplace transform

$$R_n(x, \xi_{1:n}, s_{1:n}) = \int_{\mathbb{R}^n} r_n(x, \xi_{1:n}, \tau_{1:n}) e^{-(s_1 \tau_1 + \dots + s_n \tau_n)} d\tau_{1:n} \quad (26)$$

for all $(s_1, \dots, s_n) \in \mathbb{C}^n$ such that the integral is absolutely convergent.

Remark 2 (Causal system) If the system is causal, then $r_n(x, t, \xi_{1:n}, \tau_{1:n})$ is zero as soon as τ_i ($1 \leq i \leq n$) is larger than t . Then, $\mathbb{R}^n$ can be replaced by $] - \infty, t]^n$ in (23) and by $\mathbb{R}_+^n$ in (25-26).

Remark 3 (Link with the Green's function formalism)

From remark 2, if a causal system is linear ($r_n = 0$ for $n \geq 2$), then (25) coincides with (4) where the first order kernel r_1 interprets as the standard Green's kernel g .

3.2 Interconnection laws

The interconnections laws of standard Volterra series can be generalized to the case of Green-Volterra series.

Consider two systems defined by the Green-Volterra series $\{p_n\}_{n \in \mathbb{N}^*}$ and $\{q_n\}_{n \in \mathbb{N}^*}$, respectively. Deriving the outputs and the Green-Volterra kernels of the systems described in figures 5a-c leads to the following results in the Laplace domain: For a sum of outputs $R_n(x, \xi_{1:n}, s_{1:n}) =$

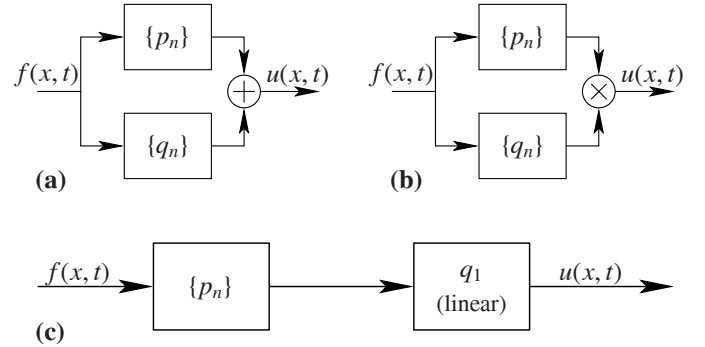


Figure 5: Sum (a), product (b), and cascade (c) of two Green-Volterra systems f .

$P_n(x, \xi_{1:n}, s_{1:n}) + Q_n(x, \xi_{1:n}, s_{1:n})$, for a product of outputs $R_n(x, \xi_{1:n}, s_{1:n}) = \sum_{m=1}^{n-1} P_m(x, \xi_{1:m}, s_{1:m}) \times Q_{n-m}(x, \xi_{m+1:n}, s_{m+1:n})$ and for a cascade with a linear system $R_n(x, \xi_{1:n}, s_{1:n}) = \int_{\Omega} Q_1(x, \xi, s_1 + \dots + s_n) P_n(\xi, \xi_{1:n}, s_{1:n}) d\xi$.

4 Closed-form solutions of the Green-Volterra kernels of the string

4.1 Problem

Let remind, the adimensionned Kirchhoff string model

$$\partial_t^2 u(x, t) + 2\alpha \partial_t u(x, t) = \left(1 + \epsilon \left[\int_0^1 (\partial_x u(x, t))^2 dx \right] \right) \partial_x^2 u(x, t) + f(x, t) \quad (27)$$

where $u(x, t)$ is the transverse displacement of the string and $f(x, t)$ a distributed excitation force. The boundary

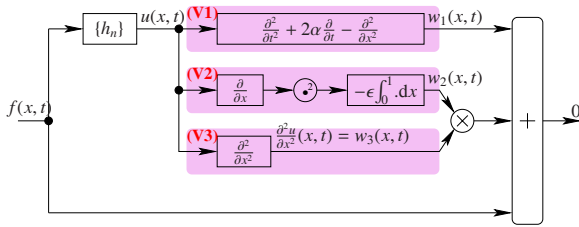


Figure 6: Equivalent block-diagrams representing Eq. (27).

conditions are Dirichlet homogeneous ($\forall t > 0, u(0, t) = u(1, t) = 0$) and the string is at rest for $t \leq 0$ ($u(x, t) = 0$).

Let be $R_n(x, \xi_{1:n}, s_{1:n})$ for $n \in \mathbb{N}^*$, the Green-Volterra kernels of the system. Using canceling system R_n is defined by

$$\Gamma^2(s_1 + \dots + s_n) R_n(x, \xi_{1:n}, s_{1:n}) - \partial_x^2 R_n(x, \xi_{1:n}, s_{1:n}) = E_n(x, \xi_{1:n}, s_{1:n}) \quad (28)$$

with the following boundary conditions $\forall (s_{1:n}) \in \{0; 1\}(\mathbb{C}_0^+)^n$

$$R_n(x = 0, \xi_{1:n}, s_{1:n}) = R_n(x = 1, \xi_{1:n}, s_{1:n}) = 0 \quad (29)$$

and

$$E_1(x, \xi, s) = \delta(x - \xi) 1(s) \quad (30)$$

$$E_n(x, \xi_{1:n}, s_{1:n}) = \epsilon \int_{\Omega} \sum_{p+q+r=n} \partial_x H_p(x, \xi_{1:p}, s_{1:p}) \partial_x H_q(x, \xi_{p+1:p+q}, s_{p+1:p+q}) \partial_x^2 H_r(x, \xi_{p+q+1:n}, s_{p+q+1:n}) \quad (31)$$

Problem (28) to (31) is similar to (5) in which $U(x, s) \mapsto R_n(x, \xi_{1:n}, s_{1:n})$, $F(x, s) \mapsto E_n(x, \xi_{1:n}, s_{1:n})$ and $\Gamma(s) \mapsto \Gamma(s_1 + \dots + s_n)$.

The solution is (as for problem (5)) defined in Eqs. (8) and (9).

For $n = 1$,

$$R_1(x, \xi, s) = G(x, \xi, s) \quad (32)$$

defining the same problem as in section 2.2.

For $n \geq 2$

$$R_n(x, \xi_{1:n}, s_{1:n}) = \int_{\Omega} G(x, y, s_1 + \dots + s_n) E_n(y, \xi_{1:n}, s_{1:n}) dy \quad (33)$$

defining the same problem as in section 2.3.

This is the solution of the linear Eq. (5). The Green function of the string displacement is $G(x, y, s)$ which is also the first order Volterra kernel. Let now compute higher order Volterra kernels.

5 Modal decomposition and realization

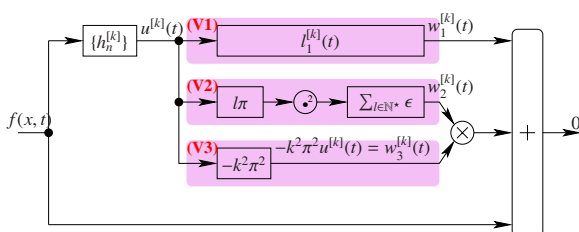


Figure 7: Equivalent block-diagrams representing Eq. (27).

For all $n \in \mathbb{N}^*$, Eqs. (28) to (31) are similar to the problem defined in Eqs. (5) to (7), it can be therefore

decomposed on a modal basis with the following definition valid in the set $L^2(\Omega)$

$$R_n(x, \xi_{1:n}, s_{1:n}) = \sum_{k \in \mathbb{N}^*} R_n^{[k]}(\xi_{1:n}, s_{1:n}) e_k(x) \quad (34)$$

and

$$R_n(x, \xi_{1:n}, s_{1:n}) = \sum_{(k,l) \in (\mathbb{N}^*)^2} R_n^{[k][l]}(s_{1:n}) e_l(x) \quad (35)$$

with $e_k = \sqrt{2} \sin(k\pi x)$. Using the orthogonality of the modes, $R_n^{[k][l]} = 0$ if $k \neq l$, we will therefore define $R_n^{[k]} = R_n^{[k][k]}$.

Green-Volterra kernels $R_n^{[k]}$ are defined by

$$(s^2 + 2\alpha s) R_n^{[k]}(s_{1:n}) + k^2 \pi^2 R_n^{[k]}(s_{1:n}) = E_n^{[k]}(s_{1:n}) \quad (36)$$

with

$$E_1^{[k]}(s) = 1(s) \quad (37)$$

$$E_n^{[k]}(s_{1:n}) = -\epsilon k^2 \pi^4 \sum_{p+q+r=n} \sum_{l \in \mathbb{N}^*} l^2 R_p^{[l]}(s_{1:p}) R_q^{[l]}(s_{p+1:p+q}) R_r^{[k]}(s_{p+q+1:n}) \quad (38)$$

Problem (36) to (38) is solved by Eq. (11) in which $U(x, s) \mapsto R_n^{[k]}(s_{1:n})$, $F_k(s) \mapsto E_n^{[k]}(s_{1:n})$ and $\Gamma(s) \mapsto \Gamma(s_1 + \dots + s_n)$.

For $n = 1$

$$R_1^{[k]}(s) = G_k(s) \quad (39)$$

defining the same problem as in section 2.2.

For $n \geq 2$

$$R_n(s_{1:n}) = G_k(s_1 + \dots + s_n) E_n^{[k]}(s_{1:n}) \quad (40)$$

defining the same problem as in section 2.3.

Using previous work (cf. section 2.3.4), the Green-Volterra kernels modal projections can be identified to the same structure presented in figure 3. Indeed, for $n = 1$, the Green-Volterra kernel is the Green function (cf. Eqs. (10)- (11)), and for $n \geq 2$, the kernels consists in sums and products of the output of this already known Green function.

Therefore, the simulation will be performed using the structure presented in figure 8. The only difference with the previous work, is the input of the system, where the force $f(x, t)$ is decomposed on the modal basis using a scalar product for each mode.

6 Realizations and discussion

Figure 9 represent the time response of the string at an observation point $x = 0.57$, for an excitation force $f(x, t) = (|x - x_0| < l) \cos(2\pi(x+t)) * f_{max} * t / T_{force}$ where $x = 0.35$, f_{max} is the maximum amplitude of the mass force, $T_{force} = 0.01s$ and $l = 0.04$.

The sampling frequency is $f_s = 44100 \text{ Hz}$ and the number of modes is $K = 20$. The same conclusions can be made in comparison with [1], i.e. the approximation at a given order of nonlinearity (here $N = 5$) is valid until a maximum of force amplitude, where the nonlinear response have the same magnitude than the linear one. Without a calculation of the convergence radius, this is a good observation, to define a valid range of excitation force.

Figure 10 presents the main advantage of Green-Volterra kernels, in comparison with previous works, knowing, a

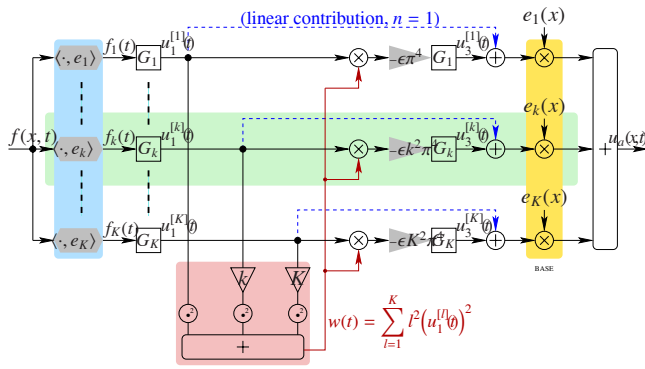


Figure 8: Block-diagram of an $o(\epsilon)$ -simulation of Eq. (1) based on Green-Volterra kernels with K modes: the excitation force $f(x, t)$ is completely unknown to the system and is decomposed on the modal basis using a scalar product.

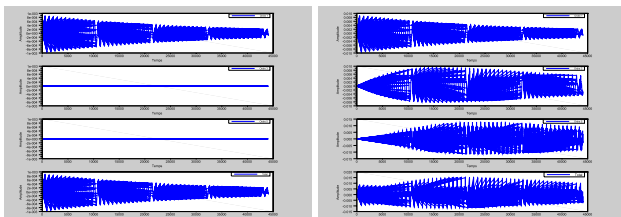


Figure 9: Results of the figure 8 simulation for $F_{max} = 500 \text{ N/Kg}$ and $F_{max} = 5000 \text{ N/Kg}$. From top to bottom: linear, third order and fifth order responses, and their sum.

simulation with an input $f(x, t)$ allowing the force to vary in time and space during the simulation. In this case, excitation force is modulated by $\cos(2\pi x t)$, thus moving the spatial position of the force with time.

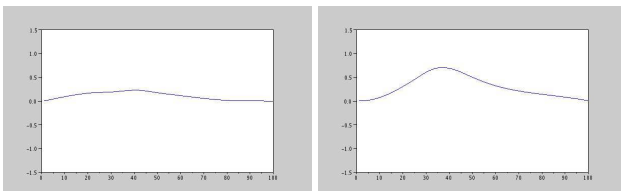


Figure 10: String deformed by the space moving force $f(x, t)$ at time t_0 and $t_1 > t_0$.

7 Conclusion

This paper has shown the calculation of Green-Volterra kernels for the Kirchhoff string model. This formalism allows to use the Green's function properties in the dynamics of a weakly nonlinear problem. The main result of the computation is the space variation of the excitation force applied to the string.

This generalization of an excitation force $f(x, t)$, is a requirement to work on another kind of problems. Contact problems for deformable solids are a very good example of the need of a variable space distribution of the force. This paper is a possible answer for such issues.

References

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