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Nonlinear guided waves to characterize damage in glass fiber reinforced plastics (FRPC)

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Sensitivity of nonlinear acoustic methods to the presence and the evolution of micro-damage has been proven in various studies on a wide range of materials (composites, concrete, rocks, etc.). This sensitivity is due to the relationship existing between the elastic properties of the above-mentioned materials and the created strain during the passage of the acoustic disturbances. Single mode nonlinear resonance experiments were proposed to explain the decrease in resonance frequency as well as the quality factor as a function of the dynamic strain. Thereby, the hysteretic parameters corresponding to the elastic modulus as well as damping are determined for a single frequency. In this work, we propose a guided wave approach to characterize GFRP composite samples taken at intact as well as damaged states. The changes in the nonlinear hysteretic parameters are observed by changing the order the excited bending resonance. In addition, the velocity dispersion of the generated guided waves (flexural waves) has also been determined and followed at different strain levels.

1 Introduction

Nonlinear acoustics methods are in intense development for nondestructive testing and evaluation on a wide range of materials (composites, concretes, rocks, bone, etc.) [1-4]. These methods present a high sensitivity compared to linear methods (velocity, attenuation). Indeed, damaged materials and consolidated heterogeneous materials, exhibit a linear or classical nonlinear behavior involving quadratic and / or cubic elastic parameters, which are considered in the context of the elasticity theory of Landau, as long as they are excited at low strain rates. At higher strains (above ~10^{-6}) these materials are non classical nonlinear and exhibit a "new" behavior usually considered as an out of equilibrium state [5-8]. Considering this non-classical behavior, different works were conducted in order to understand it, including nonlinear resonances, harmonics generation, self demodulation, transfer of modulation, self-action, etc. [9-11]. It has been then observed that nonlinear parameters are more sensitive to the presence of damage than linear elastic parameters especially at early damage states. In resonance experiments, two hysteretic nonlinear parameters are assessed from the downshift of the resonance frequency α_f as well as the diminution of the quality factor α_Q with increasing strain amplitude. These hysteretic parameters are often considered in the case of a single resonance mode neglecting the possibility of having any dispersion when higher order resonances are involved. The purpose of this paper is to report an experimental method which describes the dispersion of the nonlinear hysteretic parameters in the case of bending waves and guided waves propagating in GFRP plates taken at intact and damaged states [12].
In the first section, we recall the theory of bending waves. Then, the useful principles of nonlinear elasticity of micro-damaged materials are presented. Finally, the studied materials and the experimental procedure are described and right before the experimental results.

2 Theory of flexural and guided waves

The equation of transverse motion resulting from a bending action is based on the Euler-Bernoulli classical beam theory [13-15]. The governing equation is written in this form

$$EI_{Gz}\frac{\partial^4 v}{\partial x^4}+\rho S\frac{\partial^2 v}{\partial t^2}=0 \qquad (1)$$

Euler-Bernoulli beam theory admits two important approximations: (a) the deformations of the cross section due to the shear and rotary-inertia effects are neglected, and (b) applying the method of separation of variables to find the arrow. To do this, we assume that the motion is harmonic (sinusoidal in the time)

$$v(x,t)=V(x)\,G(t) \qquad (2)$$

Where

V(x): is the amplitude of vibration along axis x.

$$G(t)=G_1\sin(wt)+G_2\cos(wt) \qquad (3)$$

The dynamic beam equation (1) becomes

$$EI_{Gz}G(t)\frac{\partial^4 V}{\partial x^4}+\rho SV(x)\frac{\partial^2 G}{\partial t^2}=0 \qquad (4)$$

and

$$\ddot{G}(t)=-w^2G(t) \qquad (5)$$

The differential equation giving the amplitude of the deformation *V(x)* at angular frequency *w* is given by

$$EI_{Gz}\frac{\partial^4 V}{\partial x^4}-\rho Sw^2V=0 \qquad (6)$$

The amplitude equation is given by the solution of 4th order differential equation. It can be therefore written in this form

$$V_i(x)=A\cosh(\beta_i x)+B\sinh(\beta_i x)+C\cos(\beta_i x)+D\sin(\beta_i x) \qquad (7)$$

$$\beta_i=\sqrt[4]{\frac{\rho Sw_i^2}{EI_{Gz}}} \qquad (8)$$

β_i is the wave number, *E, ρ,* et *S* are respectively the Young modulus of the material, the density of the material and the air of the cross section of the bending beam. The moment of inertia of the section S, is:

$$I_{Gz}=\frac{bh^3}{12} \qquad (9)$$

There are an infinite number of values that allow β_i responding to the solution of the system. Each value β_i (i=1,2,3,4) corresponds to a vibration mode.
The natural frequencies (resonance frequencies) are obtained from the wave number. It is a question of imposing on the model of the beam (equation of amplitude) the conditions which correspond to the generated physical effect [13].
The appropriate frequency of every mode of flexion is given by this relation:

$$f_i = \frac{1}{2\pi}\alpha_i^2\sqrt{\frac{EI}{\rho S L^4}} \qquad (10)$$

$\alpha_i = \beta_i L$ are the roots of the equation for eigenfrequencies (Eq.11) defined by the boundary conditions in the case of a beam with free ends, L is the length of the beam.

$$\cosh(\beta_i L)\cos(\beta_i L) = 1 \qquad (11)$$

The bending wave velocity is given by:

$$v_{flexion}\,(\text{mode i}) = \frac{w_i}{\beta_i} = \sqrt[4]{\frac{E\pi^2 h^2 f_i^2}{3\rho}} \qquad (12)$$

This equation shows the dispersive nature of bending wave (the flexural wave velocity depends on its frequency). Flexural waves can also be described by a more advanced theories, such as Rayleigh theory or Timoshenko beam theory. Timoshenko model incorporates both the rotary-inertia and shear deformation that affects the natural bending frequencies. These two effects tend to reduce the resonance frequency calculated due to the growth of the inertia and flexibility of the system. Then the governing equation of motion developed by Timoshenko [17-20] given by:

$$EI_{Gz}\frac{\partial^4 v}{\partial x^4} - \rho I_{Gz}\left(1+\frac{E}{kG}\right)\frac{\partial^4 v}{\partial x^2 \partial t^2} + \frac{\rho^2 I_{Gz}}{kG}\frac{\partial^4 v}{\partial t^4} + \rho S\frac{\partial^2 v}{\partial t^2} = 0 \qquad (13)$$

But, if the effect due to the shear distortion is the neglected, the Rayleigh beam equation [21] arises:

$$EI_{Gz}\frac{\partial^4 v}{\partial x^4} - \rho I_{Gz}\frac{\partial^4 v}{\partial x^2 \partial t^2} + \rho S\frac{\partial^2 v}{\partial t^2} = 0 \qquad (14)$$

When we compare the values of wave velocity resulting from different theories Bernoulli, Timoshenko and Rayleigh as a function of the wave number (see figure 1), we note that for these three models, the values of bending speed are equivalent for small values of wave number. Thus the use of one of these theories to plot the dispersion curves of flexural at low frequency is the same. The literature shows that it is possible to generate an anti-symmetric Lamb mode A_0 from bending waves for low frequency values in isotropic structures such as aluminum.

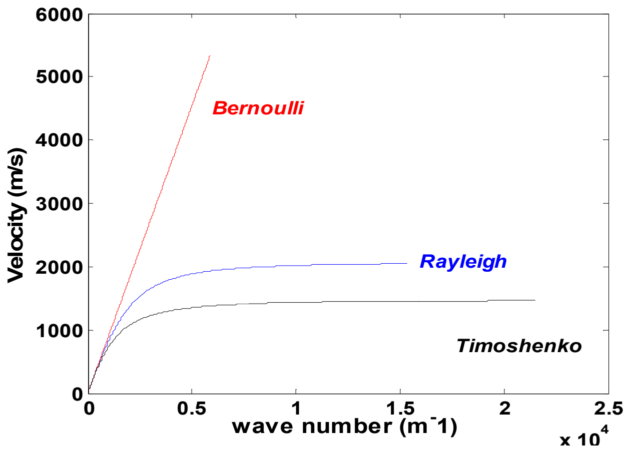

Figure 1: Dispersion relations from Timoshenko, Rayleigh and Bernoulli-Euler beam theories.

We superimposed the dispersion curves of flexural waves and Lamb waves in the case of polymer based composite with 2.52mm thickness. Indeed, Figure 2 shows that these two types of waves have the same phase velocities for frequencies below ~29 KHz. This value corresponds to the frequency of separation of both waves.

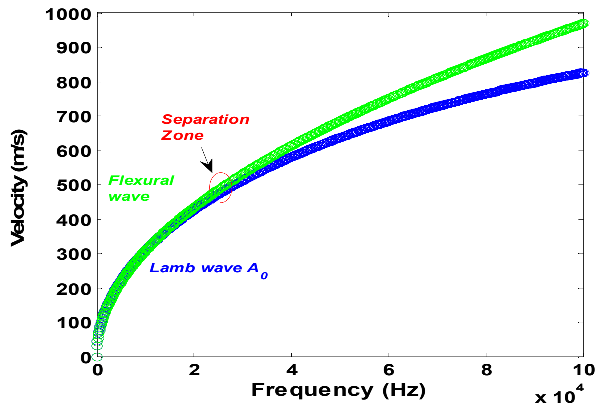

Figure 2: Dispersion curves of the bending waves and Lamb waves for a polymer based composite thickness 2.52mm.

3 Nonlinear dynamic elasticity: definition of hysteretic parameters

The behavior of nonlinear mesoscopic elastic materials is described in many articles [22-26]. Recent experiments performed on rock samples showed that for low strain amplitudes (10^{-6} and lower), nonlinear mesoscopic elastic materials behave classically [27]. However, for strain amplitudes greater than 10^{-6}, the classical theory of nonlinear elasticity [5] is unable to describe the elastic behavior of nonlinear mesoscopic materials [28,29]. Microcracks, mesoscopic flexible inclusions in a matrix, etc. are responsible of the nonlinear behavior due to the presence of discrete memory and hysteresis in the stress-strain relationship. The physical mechanisms underpinned are not yet fully understood but Guyer and McCall proposed a phenomenological description of these non linear effects [30, 31], considering the material as a set of hysteretic units that can have only two stable states: open unit and closed unit. Each hysteretic unit can be described by two pairs of variable: (σ_0, σ_c) and $(\varepsilon_0, \varepsilon_c)$ are respectively the stress and strain in the closed and open states. Hysteretic units can be recorded in the space of Preisach-Mayergoyz (PM space), which keeps track of the status (open, closed) of all hysteretic units. Knowing the history in PM space allows establishing the equation of state [31, 32]:

$$\sigma = \int K(\varepsilon, \dot{\varepsilon})d\varepsilon \qquad (15)$$

σ is the stress, ε is the strain is the strain rate and K is the elastic modulus of the $\dot{\varepsilon}$ material.

From the equation 15, we can have the expression of the elastic modulus of the material, by calculating the mean modulus on a cycle:

$$K = K_0(1 + \beta\varepsilon + \delta\varepsilon^2 + \ldots) - \alpha(\varepsilon, \dot{\varepsilon}) \qquad (16)$$

where, K_0 is the linear elastic modulus, ε is the induced strain, β and δ represent the classical quadratic and cubic nonlinear parameters, respectively, which can be developed as a combination of second, third, and fourth order elastic constants [5, 33], α is the nonlinear hysteretic parameter.

In Eq. (16) classical and hysteretic nonlinear behaviors are clearly differentiated. Indeed in the case of nonlinear hysteretic material, Eq. (16) shows that at a certain level of deformation the elastic modulus K begins to decrease.

A decrease in the compression modulus generates a decrease of resonance frequency, and hence a decrease of the flexural wave velocity or lamb wave velocity V_{A0}. The flexural wave velocity is written in the case of nonlinear hysteretic material as:

$$v_{flexion}(\varepsilon)=v_{A_0}(\varepsilon)=\left(\frac{E(\varepsilon)\pi^2h^2}{3\rho}\right)^{\frac{1}{4}}\sqrt{f(\varepsilon)} \qquad (17)$$

In recent simulations, we have proved that Young modulus can be kept constant in the fourth root of the equation and the change in velocity is mainly due to the resonance frequency.

In most of the experimental observations, the hysteretic nonlinear behavior observed at strain amplitudes above approximately $\varepsilon \approx 10^{-6}$ during nonlinear resonance experiments exhibits the same trends [34]. Two parameters are defined in the literature corresponding to the proportionality coefficients. The parameter of hysteretic elastic nonlinearity α_f is such that

$$\alpha_f = -\frac{\Delta f}{f_0 \varepsilon_A} = \frac{f(\varepsilon_A)-f_0}{\varepsilon_A} \qquad (18)$$

Where $f(\varepsilon_A)$ the resonance frequency at the detected amplitude ε_A, f_0 is the resonance frequency at infinitely low strain amplitude and Δf is the shift of the resonance frequency caused by the increasing strains.

The parameter of hysteretic dissipative nonlinearity α_Q is defined as:

$$\alpha_Q = -\frac{\Delta(\frac{1}{Q})}{\Delta\varepsilon_A} = (\frac{1}{Q_0}-\frac{1}{Q(\varepsilon_A)})\frac{1}{\Delta\varepsilon_A} \qquad (19)$$

where, Q_0 is the quality factor of the resonance at infinitely low strain amplitude and $Q(\varepsilon_A)$ the quality factor at strain amplitude ε_A. When the parameter α is determined for different types of resonant modes (compression, bending), we still need to know it changes as a function of the applied frequency. In this work, we propose a general experimental method to identify and track the hysteretic nonlinear parameters related to the bending modes in the case of a polymer-based composite. This was done under guided waves conditions.

4 Polymer based sample and damage

The studied composite is a glass fiber reinforced polyester resin, whose fibers are all oriented in the direction $\vec{e}_1$ (see figure 4). The mechanical properties of the laminate beam of length 180mm, width 20mm and thickness 2.52mm are given by the table1.

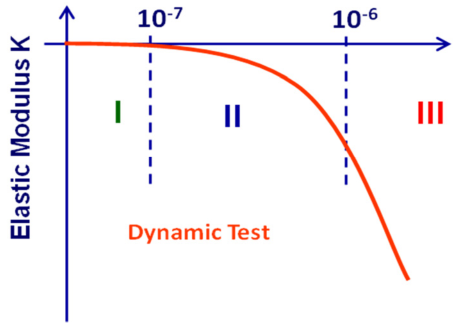

Figure 3: Elastic behavior of hysteretic nonlinear materials: (I) linear elastic range, (II) classical nonlinear elastic range, and (III) hysteretic nonlinear elastic range

Table 1: Mechanical properties of the unidirectional composite.

$\rho(kg)$	$V_T(m/s)$	$V_L(m/s)$	$E(GPa)$	ν
1492 ± 0.02	1337 ± 0.03	1337 ± 0.03	6.39 ± 0.05	0.21 ± 0.02

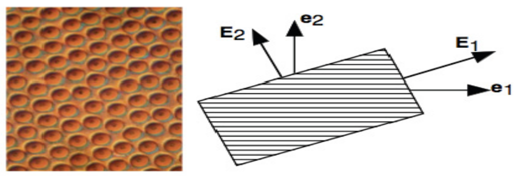

Figure 4 : Axes of the transverse isotropic composite.

In this work we are interested in the characterization of the material in the intact and damaged state. The composite is damaged with a classical quasi-static three point bending technique. In this case, the created damage can be a fiber breakage, fiber/matrix debonding, and matrix micro-cracking (see figure5).

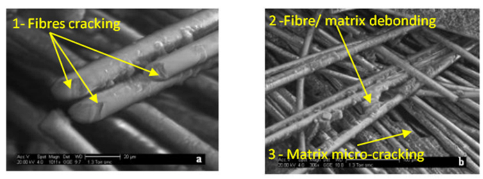

Figure 5 : The three main types of damage the composite

Resonance experiments are carried out on the same sample in order to follow the effect of damage on the excited resonance bending modes.

5 Experimental device and linearity study

The schematic of the experimental set-up is presented in figure 6. It allows to determine the phase velocity dispersion curves of antisymmetric Lamb mode A_0 at the intact and damaged states. Besides, it allows following the evolution of these phase velocity dispersion curves as a function of the induced strains.

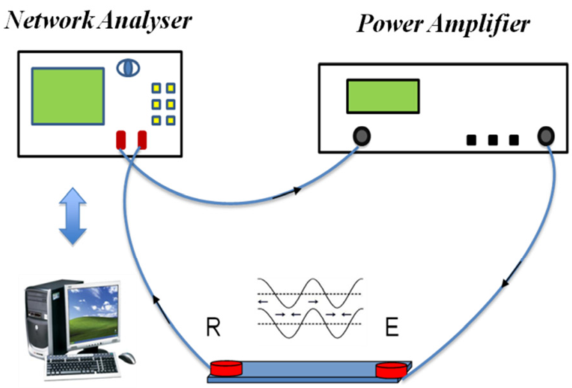

Figure 6: Experimental setup used for nonlinear measurements

A gain-phase analyzer *Stanford Reasearch Systems SR785* generates a swept-sine signal defined by a start and end frequencies ranging from 1 KHz to 20 KHz. The excitation signal is then amplified at a constant rate 46 dB using a wideband power amplifier (10KHz-10MHz) to excite the piezoelectric emitter transducer attached at the extremity of the composite plate to generate the desired vibrations. At the other extremity another PZT transducer having the same properties is used to receive the acoustic signals. The experimental configuration and the sample plate-like geometry (180*20*2.52 mm) favor the generation of bending resonance modes. The sample is excited around resonance bending modes at intact as well as damaged states using excitations from 10 mV to 640 mV, before amplification. In view of the experimental configuration one can only generate a single Lamb mode, which is the antisymmetric fundamental mode A_0 obtained on the base of bending waves.

The follow-up of the evolution of the transfer function of the PZT ceramics over the used frequency range, to excite amplified resonances (Figure 7), shows that the response of PZT on the material is linear. Consequently the experimental setup is linear.

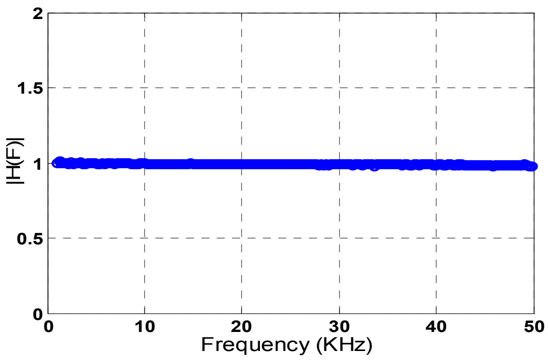

Figure 7 : Frequency response of the used piezoceramic in amplified conditions

6 Experimental results

6.1 Résonance non linéaire hystérétique

When the excitation is increased, we observe a nonlinear hysteretic behavior. The obtained set of fundamental resonance curves shows the existence of ten bending modes, where the resonance peak amplitude is expressed in terms of strain amplitude. These curves provide the hysteretic parameters α_f and α_Q that correspond to the slopes of the relative frequency shift $\Delta f / f_0$ and the change in inverse quality factor $\Delta(1/Q)$ (Table 2). The latter shows that these two parameters depend on the frequency thus the parameter α related to the resonance curves is dispersive.

Table 2: Evolution of hysteretic parameters α_f , α_Q of the polymer-based composite as a function of frequency

Flexural resonance	Δf (Hz)	α_f	α_Q
Mode 1	10	19 ± 3	1.3 ± 0.2
Mode 2	6	6 ± 1	4.0 ± 0.8
Mode 3	39.6	0.7 ± 0.1	8 ± 1
Mode 4	42	25 ± 5	9 ± 2
Mode 5	20	13 ± 2	30 ± 6
Mode 6	16	20 ± 4	83 ± 16
Mode 7	43	55 ± 11	77 ± 15
Mode 8	43	88 ± 13	162 ± 24
Mode 9	54	70 ± 14	107 ± 22
Mode 10	137	122 ± 18	106 ± 21

It should be noticed that α_f undergoes a significant change as a function of the frequency, as it changes by a factor corresponding to ~ 6. This variation is small compared to the one obtained for the dissipative parameter α_Q which increases clearly from 1.3 to 162. This means that, depending on the frequency, α_Q becomes ~ 120 times greater and hence changes 22 times faster than α_f in the same frequency range. Besides, the same set of data shows that it becomes difficult to draw a conclusion on the localization of damage as a function of nodes corresponding to the different flexural resonances. This difficulty can be observed through the different dynamics of both hysteretic parameters, whose evolutions are not monotonous as a function of the frequency. As an example, Figure 8 shows the positioning of the damage zone as a function of the generated resonances.

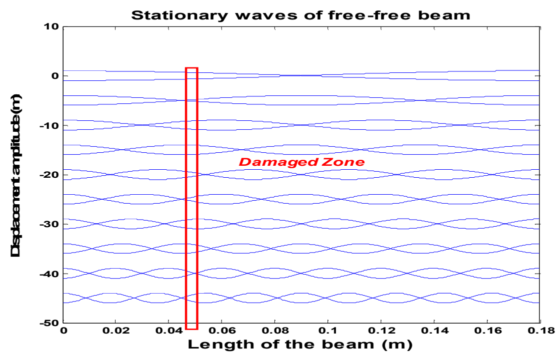

Figure 8: Superposition of the stationary waves with damage

One can observe that the maximum value corresponding to α_Q happen for the 8^{th} resonance mode. This might be explained by the fact that the damage area situated falls on the belly of the mode 8. However, if this argument is sufficient, one would find that α_f is also maximum as the positions are the same. However, this is not the case, since α_f is maximum for the 10th order resonance mode.

6.2 The dispersion of the guided waves

As explained earlier, one can determine the phase velocity of the A0 Lamb mode on the basis of the flexural resonances. Such a procedure allows determining the nonlinear hysteretic parameter associated to guided waves. This parameter is obtained by extracting the slope of the dispersion curve corresponding to A_0 mode when the drive is increased. Figure 9 shows the correspondence existing between resonances and phase velocity.

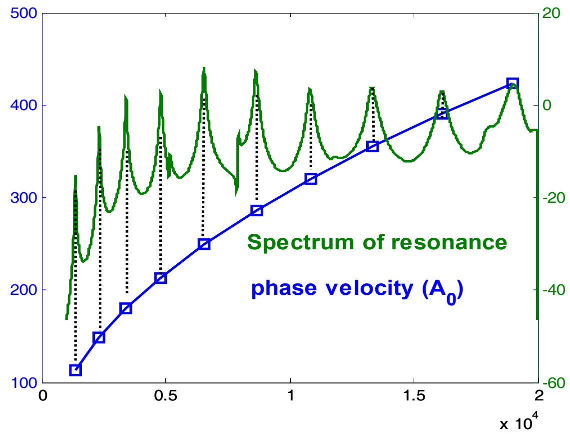

Figure 9 : Dispersion curve of the fundamental anti-symmetric Lamb mode A0 obtained from the resonance spectrum of the bending waves

We followed the evolution of the so-called speed at intact and damaged states as a function of the excitation level. We didn't notice any shift of the slope describing the change in velocity as a function of frequency at intact state. However, as the increase in the excitation level at the damaged states is accompanied by a shift of the resonance frequency (see table 2), the phase velocity values are consequently affected and so the slope of the dispersion curves.

If we study the evolution of each mode separately, by analogy with resonance experiments, we can see that it becomes possible to define a new dispersive nonlinear hysteretic parameter related to the phase velocity, called α_v, as explained in Figure 10.

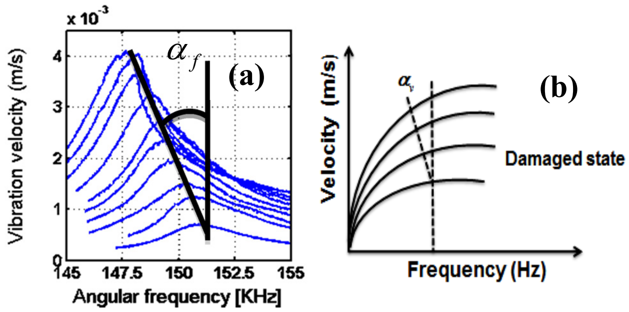

Figure 10 : Determination of nonlinear hysteretic parameter (a) elastic from resonance curves (b) dispersive from dispersion curves

The follow up of the parameter α_v as function of frequency at the damaged state shows that it is dispersive (see figure 11). Its value increases from 31m^{-1} to 88 m^{-1} as a function of damage. However, it should be notices that it remains equal to zero at the intact state.

The parameter α_v becomes increasingly important as a function of the frequency because the higher order resonances experience represent a significant frequency shift when the drive is increased. This shift of the resonance frequency causes a decrease in the velocity from the intact to the damaged states (see figure 11).

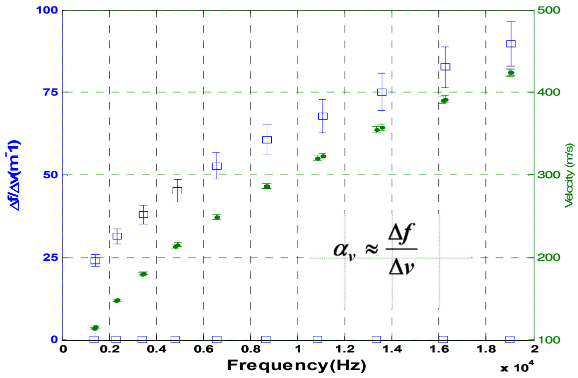

Figure 11 : Variation of α_v parameter and the phase velocity as a function of frequency at intact and damaged states

7 Conclusion and perspectives

In this work, we have developed an experimental setup to characterize the damage generated in composite plates. Hysteretic parameters α_f and α_Q revealed to be dispersive. Therefore, one should take into account this important characteristic when the nonlinear behavior is hysteretic. This observation holds also in the case where harmonics generation is studied experimentally or theoretically. Besides, we have also proved that both nonlinear parameters do not manifest any monotonous behavior as a function of frequency. The situation of the hysteretic elements near a node or far from it may be give a partial explanation. However, we think that the problem is more complex as it should include other aspects such as the anisotropy of the hysteretic zone whose damping as well as elastic properties don't seem to be symmetric as one could imagine. On the other side, the follow up of nonlinear hysteretic parameter α_v relative to the phase velocity has also shown that it is dispersive. Furthermore α_v offers an interesting opportunity to follow the evolution of hysteresis in guided waves conditions (dispersive waves), which can be in given conditions be considered as A_0 Lamb wave. Finally, it should be pointed out that for the understanding of the experimental observations we need to develop a 2D physical or phenomenological model, in order to understand the mechanisms at the origin of the unexpected experimental observations.

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