Transfer function of the atmosphere over long distances

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Propagation in the atmosphere for long range seems to be a complex phenomenon: for instance the experimental observations of signals of Concorde sonic boom to several hundred kilometers where the classical N-waves disappears completely. In fact the transfer function of the atmosphere for the near field is a pure delay (propagation time) with a gain (corresponding to the geometric attenuation).At long range this physical model is not valid. It is well known for high frequency and long range it must be adjusted for the effects of viscosity. But this model must also be corrected at low frequencies because the atmosphere is not homogeneous and therefore diffracts. Based on the linearized Euler equations , it reveals a dimensionless number g . d / a 2 (with g the gravitational constant, d the propagation distance and a the speed of sound). This number is 1 when the distance is about 5 km
The purpose of this paper is to show how the transfer function of the atmosphere changes when the propagation distance increases, assuming the latter is a stratified medium.

1 Introduction

The observations of infra-sonic propagation of impulsive distant sources show a strange behavior. In the first section we describe the main characteristics of this phenomenon which we call "rumble": lengthening of the duration increasing with the distance. In [1] it is shown these chaotic fluctuations at very low frequencies don't modify the spectrum between 1 and 10 Hz. In addition, azimuth, elevation angles and travel times are in conformity with a traditional calculation by the ray tracing method. Among various explanations, in this paper we investigate the fact that the atmosphere is a heterogeneous medium. In fact, because of the gravity, the atmosphere is a quasi-stratified medium, and the wave equation must be modified. This is done with the linearized Euler equations . In the case of constant sound velocity, the governing equations are no more wave equations but a kind of Klein-Gordon equation. Both in 3-D case (sound propagation) or in 2-D case (perturbation of supersonic body), the Green function or the atmospheric transfer function is exhibited. A dimensionless number naturally appears ($r.g/2a^2$: when it is small the waves equation is recovered but, when it is greater than 1 (when the distance is greater than 5 km which is compatible with the observations), the transfer function is dramatically different from that in homogeneous case. Thanks a stationary phase hypothesis we are able to compute a synthetic N-wave far from the source: it shows long duration and oscillations and at first sight its behavior is similar to the observations.

2 Physical Problem

The starting point of this work is the observation of the sonic boom from Concorde to a hundred miles off its trajectory (see figure 1).

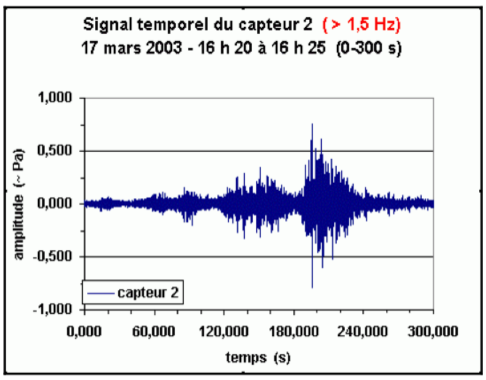

Figure 1: Distant N-wave (from [1])

As under track, a classical "N wave" can be observed (see figure 2), it is clear that far from the ground track, this N wave is indistinguishable.

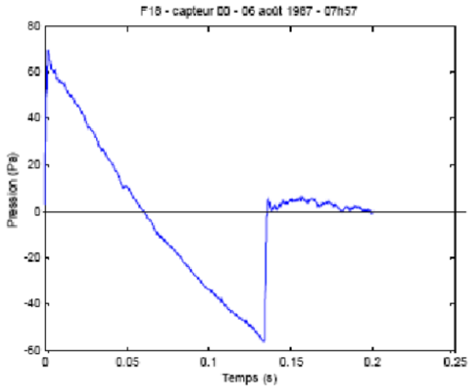

Figure 2: N-wave under track

The duration of the phenomenon (shock waves) near the plane is .1s (the length of Concorde is 60 m. and its velocity during the cruse is Mach 2). Its duration far from the ground track (30km, at the boundary of the primary carpet) can be 2 s. (see figure 3). But very far away, the signal lasts for

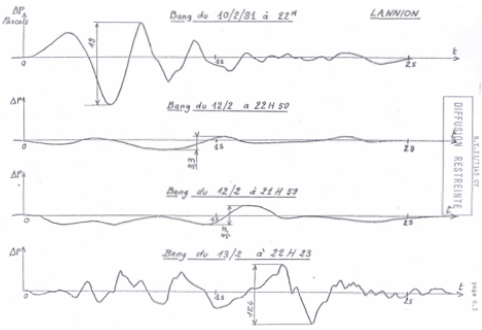

Figure 3: Smoothed N-wave

several minutes. So the question is: what are the physical mechanisms that increase the duration and cause the disappearance of the shape of the wave ? Several causes may be considered:

1. the non-linear phenomena,
2. the absorption of sound by the atmosphere,
3. diffraction of sound by the inhomogeneity of the latter,
4. turbulence,
5. or phenomena caused by caustic crossing.

The non-linear phenomena are of great importance at the beginning of propagation when the amplitude of perturbation modifies the sound speed. The shape of the pressure signal becomes an N-wave, then the amplitude decreases and nonlinear effects no longer intervene. The main effects of absorption "sweeten" the wave by killing the high frequencies but, in any case they cannot increase the duration. The caustic crossing does have a strange effect: the transmitted signal is the Hilbert transform of the incident wave. An N-wave becomes an U-wave with enhanced nonlinear effects but they cannot increase the duration. Turbulence effects are difficult to apprehend up to now. The third hypothesis is detailed in this paper. For the author, this phenomenon is very close to the rumble of thunder. During a storm one can observe that when the lightning is close to the listener, the sound of the thunder is very impulsive. Conversely when lightning is away, the signal duration lengthens and thus resembles a rumbling. It is also reported that during the First World War the noise from the front was sounded like a roar.

3 Linearized Euler Equations (LEE)

The propagation of the sound in the atmosphere is governed by LEE (see e.g. [2]). First, we have to define the fluid at rest and after to linearize around it. We assume a stratified medium without wind.Let f_0 the value of quantity f at rest.

- equation of momentum with respect of z (z is the vertical axis):
$$\partial_z P_0 = -\rho_0 g \tag{1}$$
P_0 is the pressure, ρ_0 the density et g the gravity
- given a temperature profile $T_0 = T(z)$ depending on the meteorological conditions, and thanks the law of perfect gases:
$$P = \rho R T \tag{2}$$

it is possible to compute the pressure $P_0(z)$ and density $\rho_0(z)$ profiles. To establish the equations of the fluctuations around this atmospheric state, we linearize the Euler equations. Let f_1 the fluctuation of quantity f. First we define a dimensionless density :

$$\mu = \frac{\rho_1}{\rho_0(z)}.$$

Let u, v, w the components of the velocity. If we assume the "isentropicity" of fluctuations:

$$dp_1 = a^2 d\rho_1 \quad a : \text{sound speed}$$

The LEE ca be written:

$$\begin{aligned} \partial_t u_1 + a^2 \partial_x \mu &= 0 \\ \partial_t v_1 + a^2 \partial_y \mu &= 0 \\ \partial_t w_1 + a^2 (\partial_z (\rho_0 \mu)/\rho_0) + g\mu &= 0 \\ \partial_t \mu + \partial_x u_1 + \partial_y v_1 + \partial_z w_1 + \rho_0'/\rho_0 w_1 &= 0 \end{aligned} \tag{3}$$

From now on, the index 1 can unambiguously be removed .

$$\begin{aligned} \partial_t u + a^2 \partial_x \mu &= 0 \\ \partial_t v + a^2 \partial_y \mu &= 0 \\ \partial_t w + a^2 (\partial_z \mu + (k_\rho + k)\mu) &= 0 \\ \partial_t \mu + \partial_x u + \partial_y v + \partial_z w + k_\rho w &= 0 \end{aligned} \tag{4}$$

with $k = g/a^2$ As the atmosphere is an heterogeneous (stratified) medium, it is necessary to define wave numbers $k_\rho = \rho_0'/\rho_0$ and $k_\theta = T_0'/T_0$:

$$k_\rho = \rho_0'/\rho_0 = P_0'/P_0 - T_0'/T_0 = -\gamma g/a^2 - k_\theta = \gamma k - k_\theta \tag{5}$$

This expression of is obtained thanks formula 2, 1 and the sound speed in a perfect gase:

$$a^2 = \gamma R T$$

To give an idea of order of magnitude, in the case of International Standard Atmosphere (ISA), $1/k_\theta \simeq -30km$ under $11km$ and above $k_\theta = 0$. The value of g/a^2 is $10^{-4}m^-1$ and k_ρ depend weakly on the temperature gradient. The *characteristic length of the stratification is then 10 km.* In 4 we can set:

$$\begin{aligned} \tilde{u} &= ue^{bz} \\ \tilde{v} &= ue^{bz} \\ \tilde{w} &= ue^{bz} \\ \tilde{\mu} &= \mu e^{bz} \end{aligned}$$

Then 4 is modified :

$$\begin{aligned} \partial_t \tilde{u} + a^2 \partial_x \tilde{\mu} &= 0 \\ \partial_t \tilde{v} + a^2 \partial_y \tilde{\mu} &= 0 \\ \partial_t \tilde{w} + a^2 (\partial_z \tilde{\mu} + (k_\rho + b + k)\tilde{\mu}) &= 0 \\ \partial_t \tilde{\mu} + \partial_x \tilde{u} + \partial_y \tilde{v} + \partial_z \tilde{w} + (k_\rho + b) w &= 0 \end{aligned} \tag{6}$$

If we assume a constant temperature, it is possible to define a modified wave equations for $\tilde{u}$ in this academic medium by differentiating density equation with respect to time , momenta x,y,z with respect to x,y,z:

$$\Delta\tilde{\mu} - \partial_{tt}\tilde{\mu}/a^2 - (k + 2k_\rho + 2b)\partial_z\tilde{\mu} - (k_\rho + b)(k + k_\rho + b)\tilde{\mu} = 0 \tag{7}$$

This equation is a small perturbation of classical wave equation due to the stratification. In addition, we can choose $k + 2k_\rho + 2b = 0$ and

$$\Delta\tilde{\mu} - \partial_{tt}\tilde{\mu}/a^2 + (k/2)^2\tilde{\mu} = 0 \tag{8}$$

Moreover equation 8 verifies conservation of energy. If we study the small stationary perturbation of a supersonic mobil, one obtains a similar equation (see [3]). If $M_0 = V_0/a$ is the Mach number and x axis the trajectory of the mobil, we simply replace x by t in 7 :

$$\partial_{zz}\mu + \partial_{yy}\mu - (M_0^2 - 1)\partial_{xx}\mu - (k + 2k_\rho)\partial_z\mu - (k_\rho)(k + k_\rho)\mu = 0 \tag{9}$$

We define the Prandl-Glauert Number $\beta = \sqrt{(M_0^2 - 1)}$

$$\Delta_2\mu - \partial_{xx}\mu\beta^2 - (k + 2k_\rho)\partial_z\mu - (k_\rho)(k + k_\rho)\mu = 0 \tag{10}$$

(Δ_2 is the 2-D Laplacian). This is a perturbation of the 2-D wave equation (where the time is replaced by x and the sound speed is $1/\beta$.

4 Green Function of Perturbed Wave Equation

The formulae of this section come from [4]. Equations 10 and 7 are similar. Our goal is now to solve the perturbed problems to see if the stratification hypothesis can explain the strange behavior of the signal at long range. We recall the Green functions (actually distributions) for unperturbed wave problems. For 3-D case:

$$< E, \phi >= \int_0^\infty \frac{dt}{4\pi t} \int_{r^2=c^2.t^2} \phi \, dS$$

In the 2-D case,

$$\Delta_2\mu \;-\; \beta^2 \, \partial_{xx}\mu = 0$$

the Green function is:

$$G = \frac{1}{2\pi} \frac{1}{\sqrt{x^2 - \beta^2 r^2}}$$

with now $r^2 = y^2 + z^2$ The solution of the problem (without boundary conditions) is simply the convolution of the Green function by the source h. In the case of a supersonic slender body and with a far field approximation, the solution of the pressure is given by:

$$\begin{aligned} \delta p(x-\beta r) = &\quad p_0 \frac{\gamma {M_0}^2}{\sqrt{2\beta r}} F(x-\beta r) \\ F(x) = &\quad 1/2\pi \int_0^x \frac{A''(s)}{\sqrt{x-s}} ds \end{aligned} \tag{11}$$

with F: Whitham function and A the area profile of the body. First we simplify the problem in setting $G = e^{bz}\tilde{G}$. on a:

$$\begin{aligned} \partial_z G = &\quad e^{bz}(\partial_z\tilde{G} + b\tilde{G}) \\ \partial_{zz} G = &\quad e^{bz}(\partial_{zz}\tilde{G} + 2b\partial_z\tilde{G} + b^2\tilde{G}) \end{aligned} \tag{12}$$

As G verifies (in 2-D case) ,

$$\Delta_2\mu - \partial_{xx}\mu\beta^2 - (k + 2k_\rho + 2b)\partial_z\mu - (k_\rho + b)(k + k_\rho + b)\mu = 0$$

So $\tilde{G}$ verifies:

$$\begin{aligned} \Delta_2\tilde{G} - \partial_{xx}G\beta^2 - (k + 2k_\rho + 2b)\partial_z\tilde{G} \\ -(k_\rho + b)(k + k_\rho + b)\tilde{G} = \\ e^{-bz}\delta(x)\delta(y)\delta(z - z_s) \end{aligned} \tag{13}$$

In addition one has:

$$e^{-bz}\delta(z) = \delta(z)$$

In fact if ϕ is a test function,

$$< e^{-bz}\delta, \phi >=< \delta, e^{-bz}\phi >= e^{-b0}\phi(0) = \phi(0)$$

If $2b + 2k_\rho + k = 0$, the term ∂_z disappears.

$$G = e^{-(k/2+k_\rho)z}\tilde{G}$$

$$\Delta_2\tilde{G} \;-\; \partial_{xx}\tilde{G}\beta^2 \;-\; \xi_0^2)\tilde{G} \;=\; \delta(x)\delta(y)\delta(z) \tag{14}$$

with:

$$\xi_0 \;=\; \frac{g}{2a^2} = k/2^1 \tag{15}$$

The problem is now to find the Green function of a "Klein-Gordon" equation in 2-D and 3-D case. For 3-D case, if the source is harmonic in times, the new Green function is a solution of the perturbed Helmholtz problem:

$$\Delta\tilde{G} \;+\; (k^2 \;-\; \xi_0^2)\,\tilde{G} \;=\; \delta(x)\delta(y)\delta(z)$$

[1] this spatial frequency doesn't depend on temperature gradient

the solution is well-known:

$$\tilde{G} = exp(-i\sqrt{k^2 - \xi_0^2}r/4\pi r \tag{16}$$

that means the low frequencies are no more propagative but evanescent. For the 2-D Helmholtz problem,

$$\Delta_2\tilde{G} \;+\; \xi^2\tilde{G} \;=\; \delta(y)\delta(z)$$

In that case, a possible Green function [2] is:

$$-i/4H_0^2(\xi_0 r)$$

Actually as, when z is small,

$$iH_0^2(x) \simeq \frac{2}{\pi}lnz$$

the solution must converge to the Green function of the 2-D Laplace problem

$$\frac{1}{2\pi}lnr$$

. For the modified 2-D Helmholtz problem, the fundamental solution is:

$$\tilde{G} = -i/4H_0^2[\;\sqrt{(\xi^2\beta^2 - \xi_0^2)}\; r] \tag{17}$$

If a far field approximation of the Hankel function H_0^2 is used,

$$H_0^2(z) \simeq e^{i\pi/4}\sqrt{\frac{2}{\pi z}}\,e^{-iz} \quad (2\pi > arg(z) > -\pi) \quad , \tag{18}$$

one can see the same behavior as in 3-D case i.e. if

$$\|\beta\xi\| \leq \|\xi_0\|,$$

the wave is evanescent. If we take the Fourier transform of equation 14, the Green function is the inverse Fourier transform of:

$$\frac{1}{-f^2\beta^2 + |\xi|^2 + \xi_0^2}$$

In [5], chapter 3 sections 2-7 2-8 and 2-9 give the answer: As $(k^2+P)^{-1} = (k^2+P+i0)^{-1} - (k^2+P-i0)^{-1}$, formula (10) and (11) In 2-D case the Green function has a simple expression:

$$\tilde{G} = \frac{1}{2\pi}\frac{cos(\xi_0\sqrt{x^2/\beta^2 - r^2})}{\sqrt{x^2 - \beta^2 r^2}} \tag{19}$$

When $\xi_0 \to 0$, the expression of $\tilde{G} \to$ the Green function of 2-D wave equation:

$$\frac{1}{2\pi}\frac{1}{\sqrt{x^2 - \beta^2 r^2}} \tag{20}$$

For the 3-D case the expression of Green function is:

$$\frac{1}{8\pi}J_{-1}(\xi_0 c\sqrt{t^2 - r^2/c^2})\frac{\xi_0}{\sqrt{t^2 - r^2/c^2}}$$

(J is the Bessel function).

In the 2-D case, a far-field expression of 20 is:

$$\frac{1}{2\pi\sqrt{2\beta r}}\frac{1}{\sqrt{x - \beta r}} \tag{21}$$

[2] as in equation 16, this choice is done in order to be compatible with the usual definition of Fourier and Laplace transforms and $p = i\omega + \epsilon$ with $\epsilon \geq 0$

If the same approximation is done in 19

$$\tilde{G} = \frac{1}{2\pi\sqrt{2\beta r}} \frac{cos(\xi_0\sqrt{\frac{2r}{\beta}}\sqrt{x-\beta r})}{\sqrt{x-\beta r}} \quad (22)$$

If we take the Fourier transform of 22 we obtain the expression 19 where the far-field approximation 18 of H_0^2 is done and

$$\sqrt{(\xi^2\beta^2 - \xi_0^2)} \simeq \xi\beta(1 - \frac{\xi_0^2}{2\xi^2\beta^2}) = \xi\beta - \frac{\xi_0^2}{2\xi\beta}$$

Figures 4 and 5 give the shape of the Green function for different ranges. When the latter is strongly oscillating, it is

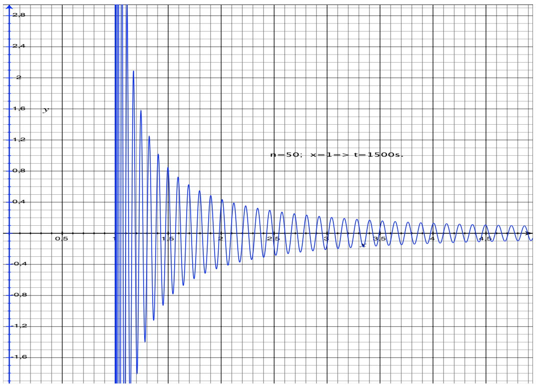

Figure 4: Green function of perturbed wave equation in the 2-D case

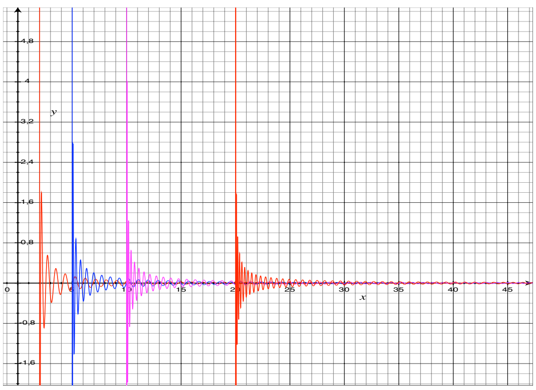

Figure 5: Green function of perturbed wave equation (2-D) for different ranges

understood that the shape of the source is completely destructured, and therefore it is impossible to recognize the original signal.

5 Synthetic Far-Field N-Wave

(see [3, 6] for details) To obtain the shape of a N-wave at long distance we have to compute the convolution of the Green function of the modified wave problem by the N-wave: we do that by multiplying the Fourier transforms of both functions and taking the inverse Fourier transform. If we assume the distance of propagation is great, a stationary phase approximation can be used in the inverse Fourier transform. The FT of an N-wave of lenth L (assumed to be odd) is:

$$\mathcal{N}(\xi) = iPmax(\frac{cos(\xi L)}{\xi L} - \frac{sin(\xi L)}{(\xi L)^2})$$

To be more realistic, we "kill" the high frequencies to simulate absorption in multiplying by the filter:

$$F(\xi) = exp(-\alpha r\xi^2)$$

The FT of a smoothed N-wave is shown in figure6.

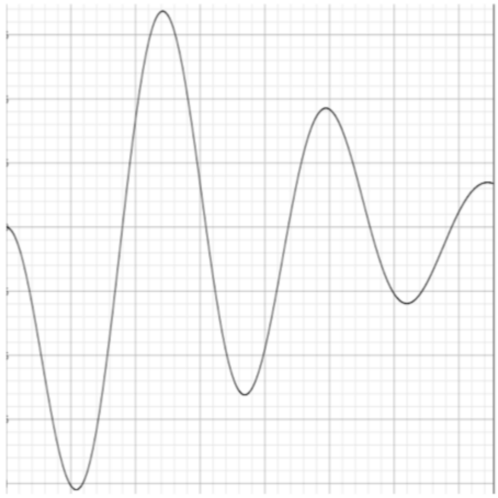

Figure 6: smoothed N-wave FT

The "temporal" signal is given by:

$$S(x) = \frac{1}{2\pi}\sqrt{\frac{2}{\pi}}e^{-i\pi/4}\int_{-\infty}^{+\infty}\frac{exp[i\phi(\xi)]}{r^{1/2}(\beta^2\xi^2-\xi_0^2)^{1/4}}\mathcal{N}(\xi)\,d\xi \quad (23)$$

$$\text{with } \phi(\xi) = x\xi - r\sqrt{\beta^2\xi^2 - \xi_0^2} \quad (24)$$

If we set

$$\xi = \frac{\xi_0}{\beta}\eta$$

then

$$\phi(\xi) = \xi_0\,[x\eta/\beta \; - \; r\sqrt{\eta^2 - 1}]$$

the phase is stationary when:

$$\frac{x}{\beta} \; - \; r\frac{\eta}{\sqrt{\eta^2-1}}$$

We define:

$$s = \frac{x}{\beta r} \quad \text{so} \quad s = \frac{\eta}{\sqrt{\eta^2-1}}$$

then

$$\hat{\eta} = \frac{s}{\sqrt{s^2-1}}$$

(remark the formulae are involutive). The corresponding phase is :

$$\phi(\hat{\eta}) = \xi_0\, r\sqrt{s^2-1}$$

It is now possible to understand how the temporal signal is built: the resulting signal is *the impulse response (Green function) modulated by the Fourier transform of the N wave from high frequencies N to low given by the relation between x and ξ obtained in "stationarizing" the phase* (to compare with figure 1)

6 Conclusions and Perspectives

To explain *the rumble phenomenon* (abnormal duration and oscillations at long distance of propagation), several hypotheses may be envisaged:

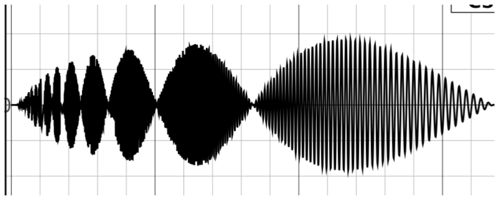

Figure 7: Synthetic N-wave

- Non-linear effects
- Dissipative processes
- Diffraction effects due to the gravity
- Turbulence
- Caustics crossing

If the two first hypothesis cause a longer duration of an impulsive source, this duration cannot be sufficient to explain the observations at long distance. Otherwise the diffractive effects due to the gravity seem to be the best candidate. In the academic (but realistic) case of a constant sound velocity where the medium remains stratified, we obtained the transfer function of the atmosphere, which is nothing but the Green function of the *stratified Helmholtz equation*. This function is approximatively equal to the usual free-space Green function when a dimensionless constant ($kr/2$ with $1/k \simeq 10km$) is small. But one has a dramatic different behavior when it becomes greater than 1, which seems to agree with observations. To be more realistic and to obtain multiple arrivals and caustics, it is necessary to take in account gradients of temperature, wind profiles, and boundary condition at the ground. To do that, one can envisage:

- Numerical simulations of L.E.E, but as the domain contains a important number of wavelength, they will be rapidly heavy.
- As the medium, atmosphere is quasi 1-dimensional, one can obtain ordinary differential equations in z (instead of partial differential equation as in the previous point) with use of adapted Fourier transform.
- to deal with the most general case, it is probably possible to compute the first term of the *WKB asymptotic development* which takes into account the multiplication term (symbol of order zero) (c.f. [7] for instance).

The main task is to build a computer code capable of taking into account not only the nonlinear effects and absorption (like in [8]) but also scattering due to heterogeneity of the atmosphere.

References

[1] G. Ménéxiadis, *Localisation de sources sonores supersoniques à partir de signaux acoustiques*, Thèse de doctorat, UNIVERSITÉ DE LA MEDITERRANEE. AIX-MARSEILLE II (2008).

[2] A. D. Pierce, *Acoustics, An introduction to its Physical Principles and Applications*, Mac Graw-Hill Book Company (1981).

[3] P. Delorme et G. Ménéxiadis, *Rumble of distant sources*, Proceedings: Acoustics Paris'08 (2008).

[4] M. Abramowitz et I. Stegun, *Handbook of mathematical Functions*, Abramowitz and Stegun (1964).

[5] I. Guelfand et G. Chilov, *Les Distributions TOME 1*, Dunod, Paris (1962).

[6] P. Delorme, *Effets diffractifs de l'atmosphére á grande distance*, Dans *10iéme congres francais d'acoustique* (2010).

[7] P. Delorme, *High-frequency parametrix for Friedrich's system*, Dans *Proceding of WAVES 07* (2007).

[8] H. Bass, R. Raspet, J. Chambers et M. Kelly, *Modification of the sonic boom wave form during propagation from the source to the ground*, J.Acoust.Soc.Am., 111(1) (2002).