Statistical model of the impulse response of a reverberant plate: application to parameter estimation and correlation analysis

E. Moulin\textsuperscript{a}, H. Achdjian\textsuperscript{a}, J. Assaad\textsuperscript{a}, F. Benmeddour\textsuperscript{a}, K. Hourany\textsuperscript{b} and Y. Zaatar\textsuperscript{b}

\textsuperscript{a}IEMN, Le Mont Houy, F-59313 Valenciennes, France
\textsuperscript{b}Université Libanaise,
emmanuel.moulin@univ-valenciennes.fr
The acoustic reverberation in a solid plate can be conveniently modeled through a nonstationary random process based on the image-sources method. The interest of such a statistical model is to allow the prediction of a general behavior (in the form of mathematical expectations) from a limited set of experimentally accessible parameters. Contrary to previous works, the model presented here takes into account the dispersive nature of plate waves (Lamb waves). Then, the relations between the average signal envelopes or correlation functions and the reverberation properties of the medium and the relative positions of the noise source, the sensor(s) and, possibly, the defect are derived. Two kinds of possible applications of these theoretical results will be illustrated: (1) parameter extraction such as source-receiver distance, structural dimensions or propagation velocities through curve-fitting; (2) imaging potentialities from ambient acoustic noise correlation.

1 Introduction

Multi-path acoustic propagation inside a closed medium contains potentially useful information about the structural properties of the medium. Means of extracting this information are currently the subject of intense research work. Powerful techniques such as time-reversal acoustics [1, 2], field correlation [3, 4, 5, 6], or diffusing acoustic wave spectroscopy [7, 8] all belong to this context.

In highly reverberant media, the low acoustic attenuation and the multiple reflections should allow both a full-scale exploration of the structural properties and a high sensitivity to the presence of a defect.

In this paper we present a time-domain statistical description, relying on a shot-noise assumption, well suited to predict the average characteristics of dispersive signals received at localised points of a reverberant medium. The basis of this model is the image-source method [9, 10], that is used to estimate the average wavepacket distributions in the signals.

First the useful theoretical relationships are derived from the statistical model. Theoretical expressions for the mathematical expectations of the squared envelopes and the squared correlation functions of signals received after reverberant dispersive propagation in the medium are obtained. Then, interesting and original possible applications of these relationships to parameter estimation and defect detection capabilities are demonstrated.

2 Theoretical developments

2.1 Statistical description of plate reverberation

Let us consider a propagation medium subject to an acoustic excitation caused by a single source emitting a signal \( s_0(t) \). If the medium is unbounded, then the signal \( s(r, t) \) received at a distance \( r \) away from the source may be characterised by its Fourier transform \( \tilde{s}(r, \omega) \) expressed as:

\[
\tilde{s}(r, \omega) = a(r) B(\omega) \tilde{s}_0(\omega) e^{-\gamma(\omega) r} e^{j k_d(\omega) r}
\]

(1)

where \( \tilde{s}_0 \) is the Fourier transform of \( s_0 \), \( B \) is the frequency-dependent excitation amplitude, \( k_d(\omega) \) is the dispersion law, \( \gamma(\omega) \) is the attenuation law and \( a(r) \) is the geometrical spreading term (\( = 1/\sqrt{r} \) in the case of two-dimensional propagation).

If the medium is bounded and with low attenuation, then the signal \( h_0(t) \) received at a given receiver \( R \) consists of the direct propagation signal \( h_0^D(t) = s(r_0, t) \) from \( S \) to \( R \), plus a reverberant part \( h_0^R \) corresponding to an infinite series of reflections on the medium boundaries:

\[
h_0(t) = h_0^D(t) + h_0^R(t)
\]

(2)

with

\[
h_0^D(t) = \sum_{i=1}^{\infty} \kappa_i s(r_i, t)
\]

(3)

where \( \kappa_i \) is the number of wavepackets coming from image-sources located at distances between \( r_i \) and \( r_i + \Delta r_i \) from the receiver. If \( \Delta r_i \) is chosen small enough, then \( \kappa_i \) is either 1 or 0.

As explained in previous works [10], the contribution of the reflections can be conveniently described as a random Poisson process (shot-noise model) of characteristic parameter \( \lambda \). Physically, \( \lambda(r) \) can be interpreted as the average density of wavepackets propagated over the distance \( r \). Hence, it is directly related to the expected value of \( \kappa_i \) in the following way:

\[
E[\kappa_i] = \lambda(r_i) \Delta r_i
\]

(4)

The expression of \( \lambda(r) \) depends on the geometry of the reverberant propagation medium and can be conveniently determined using the image-source method. This will be illustrated here in the simple case of a rectangular-shaped medium subject to 2D-propagation (single-mode plate wave).

Let us consider a rectangular plate of surface \( S \) (represented by the solid lines in Fig. 1). The images of the primary source (emitter position) are located according to a periodic pattern, whose elementary cell is the dashed-line rectangle represented in Fig. 1 [9].

![Figure 1: Image sources for a rectangular plate.](image)

Each image source will generate its own wavepacket eventually arriving at the receiver. The ones located inside a disk of radius \( r \) will then produce wavepackets propagated over distances smaller than \( r \). The number \( \Lambda(r) \) of these image sources is four times the number of elementary cells included.
in the disk. For \( r \) much greater than the plate dimensions, it corresponds approximately to the ratio of the disk surface and the plate surface:

\[
\Lambda(r) \approx \frac{\pi r^2}{S}
\]  

(5)

Then, the number of wavepackets propagated over distances between \( r \) and \( r + dr \) is \( \Lambda(r + dr) - \Lambda(r) = d\Lambda \). The wavepacket density \( \lambda \) is thus defined as:

\[
\lambda(r) = \frac{d\Lambda}{dr}
\]  

(6)

From Eq. (5), \( \lambda \) can then be estimated as:

\[
\lambda(r) \approx \beta_d r
\]  

(7)

with

\[
\beta_d = \frac{2\pi}{S}
\]  

(8)

For 2D-propagation case, the wavepacket density \( \lambda \) is thus a linear function of the propagation distance \( r \). For structure shapes other than a rectangle, Eq. (7) still applies, but \( \beta_d \) would have an expression other than Eq. (8).

### 2.2 Derivation of the ensemble-averaged envelope

The envelope of \( h^0(t) \) can be determined from the its complex analytic representation \( H^0(t) = h^0(t) + j\hat{h}^0(t) \), where \( \hat{\cdot} \) represents the Hilbert transform.

Then, from Eq. (3) the square of the envelope can be expressed as:

\[
|H^0(t)|^2 = \sum_{i=0}^{\infty} \sum_{j=1}^{\infty} |s(r_i, t) s(r_j, t) + \hat{s}(r_i, t) \hat{s}(r_j, t)|
\]  

(9)

where \( \hat{s}(r_i, t) \) is the Hilbert transform of \( s(r_i, t) \).

Separating in the double sum above the terms with \( i = j \) and the cross-terms, and taking the expected value then yields:

\[
E[|H^0(t)|^2] = \sum_i E[|s|^2] \left| s(r_i, t) \right|^2 + \sum_i \sum_{j \neq i} E[|s|] E[|\hat{s}|] \left| s(r_i, t) \hat{s}(r_j, t) \right| + \left| \hat{s}(r_i, t) s(r_j, t) \right|.
\]  

(10)

Assuming that the random variables \( \kappa_i \) and \( \kappa_j \) are independent for \( i \neq j \), remarking that \( \kappa^2 = \kappa_i \) (since it is either 0 or 1) and invoking Eq. (4) the expected values in the equation above can be simplified to:

\[
E[|s|^2] = E[|\kappa|^2] = E[\kappa_i] E[\kappa_j] = \lambda(r_i) \lambda(r_j) \Delta r_i \Delta r_j
\]  

(11)

for \( i \neq j \), and

\[
E[|\kappa|^2] = E[|\kappa_i| E[|\hat{s}|] = \lambda(r_i) \Delta r_i
\]  

(12)

Replacing the discrete sums by integrals (\( \Delta r_i \to dr_i, \Delta r_j \to dv \)) then yields:

\[
E \left[ |H^0(t)|^2 \right] = \int_0^{\infty} \lambda(r) \left| s^2(r, t) + \hat{s}^2(r, t) \right| dr
\]  

(13)

\[
+ \int_0^{\infty} \int_0^{\infty} \lambda(r) \lambda(v) |s(r, t) \hat{s}(v, t) + \hat{s}(r, t) s(v, t)| dr dv
\]

and thus

\[
E \left[ |H^0(t)|^2 \right] = \int_0^{\infty} \lambda(r) |s(r, t)|^2 dr + \left| \int_0^{\infty} \lambda(r) s(r, t) dr \right|^2
\]  

(14)

We will make here the assumptions that the excitation signal \( s_0 \) is of finite duration \( T \) and with a narrow frequency band centered at \( \omega_0 \), and that the attenuation coefficient \( \gamma(\omega) = \gamma_0 \) is approximately constant in this frequency band. Note that if actually \( s_0 \) is not narrow-band then, simply, narrow-band filtering may be applied to the received signals.

First it can be shown that, in such conditions, the second integral in the relation above is zero (see [11]), meaning that the cross-terms combination vanishes on average. Second, further developments (fully detailed in [11]) on the first integral lead to the following expression of the envelope:

\[
E \left[ |H^0(t)|^2 \right] \approx \lambda(r) a^2(r) D e^{-2i/\tau}
\]  

(15)

where

\[
D = \int_0^{\infty} v_g(\omega) \left| B(\omega) \tilde{s}_0(\omega) \right|^2 d\omega
\]  

(16)

with \( \tau = 1/(\gamma_0 v_g) \), \( r_i = v_g t_i \), \( v_0 \) is the group velocity and \( v_g = v_g(\omega) \).

In the case of 2D-propagation, \( a(r) = 1/\sqrt{T} \) and Eq. (7) yield:

\[
E \left[ |H^0(t)|^2 \right] \approx \beta_d D e^{-2i/\tau}
\]  

(17)

If \( v_g \) varies only slightly in the frequency band of the excitation, then

\[
D \approx v_g D_0
\]  

(18)

with

\[
D_0 = \int_0^{\infty} |B(\omega_0) \tilde{s}_0(\omega)|^2 d\omega
\]  

(19)

This quantity corresponds to the acoustic power injected by the source into the medium.

As will be illustrated in Sec. 3.1, comparing the theoretical expression given in Eq. (17) to measured envelopes will allow identification of physical parameters.

### 2.3 Correlation functions and sensitivity to defect

In the same way, the statistical model described above can also be used to characterise the temporal correlation functions of two received signals. This could be particularly useful when the acoustic source emits a continuous noise instead of a signal of finite duration.

The cross-correlation of the signals \( h_1(t) \) and \( h_2(t) \) received respectively at two receivers in the defect-free state is noted \( \varphi_{h_1 h_2} \). Provided the autocorrelation \( \varphi_0 = \varphi_{h_0 h_0} \) of
the emitted signal $s_0$ is of finite duration $2T$ and of narrow frequency band centered at $\omega_0$, a theoretical expression of $\varphi_{h_0h_0}(t)$ can be obtained [10]. For simplicity, this expression is given below for the case of a single receiver only (autocorrelation). Indeed, after the correlation peak (for $t \geq T$) the averaged squared autocorrelation corresponds to an exponentially decreasing function. With the additional simplifying assumptions of a constant excitation amplitude $B(\omega) = B_0$ and low dispersion, it can be expressed as:

$$E[\varphi_{h_0h_0}^2(t)] \approx A_e e^{-2/t}$$  \hspace{1cm} (20)

with

$$A_e = B_0^4 \Phi_0^2 \left( \pi v_0^2 R_d \frac{\rho_d}{\rho_0} \right) e^{4v_0/\nu_0}$$  \hspace{1cm} (21)

where $\Phi_0 = \int_0^T \varphi_s(t)^2 dt$.

The correlation function $\varphi_{h_0h_0} = \varphi_{h_0h_0} + \Delta \varphi$ in the presence of a defect can be estimated following the same kind of process [10]. The sensitivity of the correlation function is then quantified through a “defect sensitivity parameter” defined as the ratio of the energy of $\Delta \varphi(t)$, directly related to the defect’s presence, to the energy of $\varphi_{h_0h_0}(t)$, the defect-free correlation.

$$r_{12} = \frac{E[\int_0^\infty \Delta \varphi^2(t) dt]}{E[\int_0^\infty \varphi_{h_0h_0}^2(t) dt]}$$  \hspace{1cm} (22)

For a defect characterised by a linear expression $\lambda(t) = \gamma t + \varepsilon_i$ of the arrival time density of $\Delta h_i$ and after complete derivation (detailed in [10]), the following expression is obtained:

$$r_{12} \approx \kappa^2 \left[ \frac{P_2}{Q} e^{-2(t_1-t_0)/\tau} + \frac{P_1}{Q} e^{-2(t_2-t_0)/\tau} \right]$$

$$+ \kappa^4 \frac{P_2}{Q} e^{-2(t_1-t_0+t_2-t_0)/\tau}$$  \hspace{1cm} (23)

This parameter is associated to the probability to detect a given defect. Its possible use will be illustrated below (see Sec. 3.2).

### 3 Possible applications

#### 3.1 Passive parameter estimation

Eq. (17) shows that the expected envelope is characterized by a decreasing exponential function of the form:

$$E[H_0^R(t)] = A e^{-2t/\tau}$$  \hspace{1cm} (24)

with

$$A = v_0 \rho_d D_i$$  \hspace{1cm} (25)

The amplitude $A$ is related both to structural characteristics of the medium and to source properties. Identification of the value of $A$ from measured reverberant signals can then lead to interesting experimental characterisations.

This can be performed through a linear curve-fitting applied to the logarithm of the envelope averaged over a number of realisations. An illustration of this is given in Fig. 2, where numerical results obtained using a (deterministic) simulation based on the source-image method are compared to the theoretical expected envelope given by Eq. (24). The considered propagation medium is a 6-mm thick rectangular aluminum plate with lateral dimensions $2 \times 1 \text{ m}^2$. The signal $s_0$ emitted by the source is a five-cycle Hann-windowed sinusoid waveform with central frequency $f_0 = \omega_0/2\pi = 15 \text{ kHz}$. The envelope of the simulated signal received by a single receiver is shown in thin dotted blue line in Fig. 2-(a), whereas the averaged envelope over five different sensor positions is represented in Fig. 2-(b). Logically, the second (averaged) curve is closer to the theoretical expectation of the envelope shown in thick red solid-line in both figures. A simple curve-fitting applied to this one allows a very good retrieval of the parameters $A$ and $\tau$ of the theoretical curve (see the green dashed line in Fig. 2-b).

---

**Figure 2:** Simulated result and comparison to theoretical and estimated envelopes. (a) Envelope at 1 sensor position. (b) Average envelope over 5 sensor positions.

In an experimental point of view, it might not be always possible or convenient to get different realisations of received signals. Therefore it would be interesting to make the parameter identification work even on a single signal. In such case, it can be judicious to apply an integration-based technique similar to the so-called Schroeder method [12], well known in architectural acoustics. Instead of working directly on a received signal $h_0^R(t)$ or its envelope, we define the function $I_i(t) = \int_0^\infty |h_0^R(u)|^2 du$, which can be computed numerically.

Departing from Eq. (3), it can be theoretically shown that $I_i$ is also a decreasing exponential function:

$$I_i(t) = \frac{A_T}{4} e^{-2t/\tau}$$  \hspace{1cm} (26)

where $A$ is as defined in Eq. (25).

An example of application of this “Schroeder-like” method is shown in Fig. 3. The integral function $I_i(t)$ is computed from the simulated received signal (blue dashed line) and compared to the theoretical curve (red solid line) given by Eq. (26). Apart from the early-time behaviour, which is erroneously described by the statistical model, the agreement...
between both curves is very good even though a single real-
isation only is used. This remarkable behaviour means that
the time integration replaces here in some way an average
over different realisations, which is something like an ergod-
icity property. Obviously in that case, curve-fitting will also
provide results of good quality.

![Figure 3: Simulated result and comparison to theoretical
and estimated curves $I_r(t)$ for a single receiver
(Schroeder-like method).](image)

In summary, by applying either the curve-fitting process
to the averaged envelopes (when several realisations are avail-
able) or the Schroeder-like method to a single received sig-
nal, an accurate estimation of the value of $A$ may be obtained.
However, in order to be able to deduce structural character-
istics, the effective acoustic power $D_s$ injected by the source
into the medium in the considered frequency band has to be
known. Since this is highly improbable in practice, an equa-
tion will be missing to solve the problem.

What is proposed here is then to relate these source prop-
erties to the early (deterministic) part of the received signals.
If the first (direct) wavepacket is temporally separated from
the higher order ones (which is the case when the source-
receiver distance is small compared to the structure’s charac-
teristic dimensions), then a simple relationship between $D_s$
and the energy $I_0^D$ of this wavepacket (defined below) can
be used as the missing equation.

$$I_0^D = \int_{0}^{r_0+f} |H_0^D(t)|^2 \, dt \quad (27)$$

Indeed, applying the Parseval’s theorem, it can be easily
shown [11] that:

$$I_0^D = \frac{D_s}{r_0} \quad (28)$$

where $r_0$ is the source-receiver distance.

Thus, Eq. (25) and (28) constitute together the basis of an
original parameter extraction principle. Depending on the
experimental situation considered, the value of one of the
three physical parameters $v_{g_0}$, $\beta_d$ or $r_0$ can be estimated, in
addition to the acoustic power $D_s$ emitted in the frequency
band. The corresponding relationships are given below:

1. If $r_0$ and $v_{g_0}$ are known, then:

$$\begin{align*}
D_s & = \frac{r_0 I_0^D}{v_{g_0}} \\
\beta_d & = \frac{A}{r_0 v_{g_0} I_0^D}
\end{align*} \quad (29)$$

which allows an estimation of the medium dimensions
(plate surface, see Eq. 8 for the particular case of a
rectangular plate).

2. In the same way, if $r_0$ and $\beta_d$ are known, then:

$$\begin{align*}
D_s & = r_0 I_0^D \\
v_{g_0} & = \frac{A}{r_0 \beta_d I_0^D}
\end{align*} \quad (30)$$

which constitutes an estimation of the propagation ve-
locity without any time measurement.

3. And finally, if $v_{g_0}$ and $\beta_d$ are known, then:

$$\begin{align*}
D_s & = \frac{A}{v_{g_0} \beta_d} \\
r_0 & = \frac{A}{v_{g_0} \beta_d I_0^D}
\end{align*} \quad (31)$$

which allows to estimate the source-receiver distance
without time measurement and synchronization.

The third case is illustrated by promising experimental re-
results. A rectangular plate with the same characteristics as in
the simulations presented above has been used. The acoustic
excitation is provided through a pencil lead break manually
applied to the plate surface. The structural properties of the
plate being known, the values of $v_{g_0}$ and $\beta_d$ are easily accessi-
ble. The method presented above has been applied to the sig-
nal received at a single piezoelectric receiver, after band-pass
filtering by a convolution with a five-cycle Hann-windowed
sinusoid waveform with adjustable central frequency $f_0$
between 15 and 30 kHz. The receiver is separated from the
source by a distance $r_0 = 10$ cm. As shown in Fig. 4, the ap-
lication of Eq. (31) provides a fairly good estimation of the
value of $r_0$ (green circles) in the whole frequency range. Av-
eraging over the values obtained for every frequency in the
range gives $r_{0\text{avg}} = 9.5$ cm, which is not bad at all when con-
sidering the ill-calibrated excitation method and the absence
of time synchronisation between source and receiver.

![Figure 4: Estimation of $r_0$ from the measured impulse
response.](image)
3.2 Defect detection from noise correlation

To illustrate the capabilities of the correlation functions for defect detection on a practical case, we consider two sensors $R_1$ and $R_2$ and a source $S$ located at given (fixed) positions of the plate. The sensitivity of the auto- or cross-correlation to a defect of coordinates $(x, y)$ can be theoretically estimated using Eq. (23). Then, by representing the value of $r$ or $r_{12}$ in color level for each $(x, y)$ on the discretized plate surface (2D-grid), a map of the sensitivity to the defect can be established for the particular measurement configuration.

In Fig. 5, the sensitivity of the autocorrelation of the signal on a single receiver is characterised for two different source-receiver distances. The source and receiver positions are represented by a cross (+) symbol and a circle (o), respectively.

![Figure 5: Dependence of the sensitivity parameter $r$ on the defect position](image1)

What can be clearly observed from these results is that the defect detection capability is strongly related to the relative positions of the source, the receiver(s) and the defect. The obvious peak in Fig. 5-(a) shows that when the source and the receiver are close to each other, the sensitivity of the correlation function to the defect is very strong inside a small area around them and rapidly decreases as the defect position gets farther. Fig. 5-(b), corresponding to a larger source-receiver distance, exhibits a larger defect detection area, but with a lower probability of detection.

Fig. 6 shows that using two receivers and working on the cross-correlation function basically leads to the same kind of results. The additional benefit here is that positioning both receivers farther from each other (see Fig. 6-b) has the effect of extending the potential detection area, independently of the source location. Here again, the counterpart is naturally a lower sensitivity to the defect and thus detection probability.

Note that in the context of a sensor network covering the whole medium area, this effect could be compensated by the multiplication of sensors.

4 Conclusion

The results presented in this paper illustrate the possibility of extracting quantitative information from a minimal number of sensors by exploiting the statistical properties of the acoustic reverberation in a solid plate. First, using the theoretical relationship between the statistical characteristics of the envelope of the impulse response and the structural and source properties, it has been shown how parameter estimation could be achieved by associating fitted features to “ballistic” wave data. Second, the average influence of a localized defect on the correlation functions has been derived as a function of a limited number of parameters: the reverberation properties of the medium, the radiation characteristics of the damage and the relative positions of the source, the receiver and the defect. Then, the possible detection of the defect for a given source and receivers setup has been discussed.
Future work will focus on developing further this idea of combining the late and early properties of the impulse response or, in the case of a continuous noise, of the correlation functions. Taking into account not only the first, but several early reflections should increase the application potential of the method.

References


