Numerical analysis of an aeroacoustic field using Complex Variable Method

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This study deals with identification and characterization of acoustic sources in flows. Analysis of aeroacoustic noise generation and propagation often requires huge amount of data. Here, a versatile method called Complex Variable Methods (CVM) is proposed. It is a powerful tool dedicated to analyze numerical simulations by adding a small imaginary part to variables and parameters, without post-processing. Depending on the initialization of these imaginary parts, CVM provide different informations. These methods are applied to the linearized Euler's equations, and illustrated with numerical simulations showing that it can be efficiently used to get the sensibility to various parameters or to distinguish and follow the acoustic part in the total fields.

1 Introduction

The reduction of noise level for automotive vehicle passengers has become a priority for car manufacturers. A part of this noise originates from the flow. Indeed, there are different type of sources due to flows [1, 3]. In this context, it is interesting to identify and to characterize acoustic sources in flows. Then, it's important to develop some original tools aimed at analyzing and tracking acoustic waves. Following these considerations, some new applications of Complex Variable Methods (CVM) allowing the resolution of both issues are proposed in this paper.

Complex Variable Methods are mathematical methods aimed at working with complex numbers. These methods was initially introduced by Lyness and Moller [7] to compute the derivative of an analytical function. It consists in adding an imaginary part to the real variable of the function. Then, if the added imaginary part is small enough, the imaginary part of the function is directly proportional to the derivative of the function[9]. The order of precision depends only on the small parameter introduced in the imaginary part. This technique has been recently extended to numerical simulations for various applications [4, 6, 11, 12]. In these applications, it was used to compute the sensibility of a chosen parameter or variable. Briefly, it consists in changing the real variables involved in numerical simulations into complex variables by adding a small imaginary part to the real variables. Then, the great advantage of this method is that the sensibility analysis is performed during the numerical simulation by investigating the imaginary part of the solution [4, 6, 11, 12], without post-processing. Moreover, the interest of such methods relies on some particular applications for which it allows the tracking of a particular event. This last result will be emphasized in the following.

Here, CVM are tested to elucidate the acoustic wave propagation through a flow. Then after presenting the reference case, a theoretical analysis of CVM is given emphasizing its potential not only for sensibility analysis but also for tracking a particular event. Two applications of CVM are then presented demonstrating 1) their interest to perform sensibility analysis and 2) their effectiveness to track the acoustic part of flows.

In the next section, the use of complex variables to analyze the acoustic part of an aeroacoustic field is theoretically presented. Numerical applications, based on multiple acoustic sources and a flow composed by a Rankine vortex, are then performed to validate the method.

2 Sound analysis using Complex Variables Method

The purpose of this study is to analyze the propagation of acoustic waves in a flow. In order to present the method of complex variables analytically, we restrict the study to the case of a stationary flow. In this case, the acoustic part is governed by the linearized Euler's equations[2, 8]:

$$\begin{aligned}
\frac{\partial \rho_a}{\partial t} + \nabla.(\rho_a \mathbf{v}_a + \rho_0 \mathbf{v}_a + \rho_a \mathbf{v}_0) &= M,\\
\rho_0\left(\frac{\partial \mathbf{v}_a}{\partial t} + \mathbf{v}_0\nabla.\mathbf{v}_a + \mathbf{v}_a\nabla.\mathbf{v}_0\right) + \rho_a\mathbf{v}_0\nabla.\mathbf{v}_0 + \nabla p_a &= \mathbf{F},\\
\frac{\partial s_a}{\partial t} + \mathbf{v}_0\nabla s_a + \mathbf{v}_a\nabla s_0 &= 0,\\
p_a - c_0^2\rho_a - \Upsilon s_a &= 0, \quad (1)
\end{aligned}$$

where

$$\begin{aligned}
p(\mathbf{x},t) &= p_0(\mathbf{x}) + p_a(\mathbf{x},t),\\
\rho(\mathbf{x},t) &= \rho_0(\mathbf{x}) + \rho_a(\mathbf{x},t),\\
\mathbf{v}(\mathbf{x},t) &= \mathbf{v}_0(\mathbf{x}) + \mathbf{v}_a(\mathbf{x},t),\\
s(\mathbf{x},t) &= s_0(\mathbf{x}) + s_a(\mathbf{x},t),
\end{aligned}$$

are pressure, density, velocity and entropy respectively, split into a stationary part (indexed by 0) and an acoustic disturbance (indexed by a). M is a mass injection term and F is a force injection term. c_0 is the velocity of sound and $\Upsilon = \left(\frac{\partial p}{\partial s}\right)_\rho$.

In order to use Complex Variable Methods (CVM), it is first necessary to add a small imaginary part to flow variables as well as parameters that are explicitly involved in the system 1:

$$\begin{aligned}
\bar{\phi}_0(\mathbf{x}) &= \phi_0(\mathbf{x}) + i\epsilon\phi_{0I}(\mathbf{x}),\\
\bar{\phi}_a(\mathbf{x},t) &= \phi_a(\mathbf{x},t) + i\epsilon\phi_{aI}(\mathbf{x},t),\\
\bar{M}(\mathbf{x},t) &= M(\mathbf{x},t) + i\epsilon M_I(\mathbf{x},t),\\
\bar{\mathbf{F}}(\mathbf{x},t) &= \mathbf{F}(\mathbf{x},t) + i\epsilon\mathbf{F}_I(\mathbf{x},t),\\
\bar{c_0} &= c_0 + i\epsilon c_{0I},\\
\bar{\Upsilon} &= \Upsilon + i\epsilon\Upsilon_I. \quad (2)
\end{aligned}$$

where ϕ represents the different flow variables:

$$\phi = p, \rho, \mathbf{v}, s \quad (3)$$

and ϵ is an arbitrary small parameter ($\epsilon << 1$). Complex variables are denoted with an overbar. Introducing these complex variables and parameters in the system 1, one obtains a complex variables system that can be divided into two systems of equations associated with the real part and the imaginary one respectively. The first one is equal to the initial Euler equations system 1 but with a precision of ϵ^2:

$$
\begin{aligned}
\frac{\partial \rho_a}{\partial t} + \nabla.\left(\rho_a \mathbf{v}_a + \rho_0 \mathbf{v}_a + \rho_a \mathbf{v}_0\right) &= M + O(\epsilon^2),\\
\rho_0\left(\frac{\partial \mathbf{v}_a}{\partial t} + \mathbf{v}_0\nabla.\mathbf{v}_a + \mathbf{v}_a\nabla.\mathbf{v}_0\right) + \rho_a\mathbf{v}_0\nabla.\mathbf{v}_0 + \nabla p_a &= \mathbf{F} + O(\epsilon^2),\\
\frac{\partial s_a}{\partial t} + \mathbf{v}_0\nabla s_a + \mathbf{v}_a\nabla s_0 &= O(\epsilon^2),\\
p_a - c_0^2\rho_a - \Upsilon s_a &= O(\epsilon^2), \quad (4)
\end{aligned}
$$

The second one, related to the imaginary part of the whole system is:

$$
\begin{aligned}
\frac{\partial \rho_{aI}}{\partial t} + \nabla.\left(\rho_{aI}\mathbf{v}_a + \rho_{0I}\mathbf{v}_a + \rho_{aI}\mathbf{v}_0\right) &+ \\
\nabla.\left(\rho_a\mathbf{v}_{aI} + \rho_0\mathbf{v}_{aI} + \rho_a\mathbf{v}_{0I}\right) &= M_I,\\
\rho_0\left(\frac{\partial \mathbf{v}_{aI}}{\partial t} + \mathbf{v}_0\nabla.\mathbf{v}_{aI} + \mathbf{v}_{0I}\nabla.\mathbf{v}_a + \mathbf{v}_a\nabla\mathbf{v}_{0I} + \mathbf{v}_{aI}\nabla\mathbf{v}_0\right) &+ \\
\rho_{aI}\mathbf{v}_0\nabla.\mathbf{v}_0 + \rho_{0I}\left(\frac{\partial \mathbf{v}_a}{\partial t} + \mathbf{v}_0\nabla.\mathbf{v}_a + \mathbf{v}_a\nabla\mathbf{v}_0\right) &+ \\
\rho_a\left(\mathbf{v}_0\nabla.\mathbf{v}_{0I} + \mathbf{v}_{0I}\nabla.\mathbf{v}_0\right) + \nabla p_{aI} &= \mathbf{F}_I + O(\epsilon^2),\\
\frac{\partial s_{aI}}{\partial t} + \mathbf{v}_0\nabla s_{aI} + \mathbf{v}_{0I}\nabla s_a + \mathbf{v}_{aI}\nabla s_0 + \mathbf{v}_a\nabla s_{0I} &= 0, \quad (5)\\
p_{aI} - 2c_0c_{0I}\rho_a - c_0^2\rho_{aI} - \Upsilon_I s_a - s_{aI}\Upsilon &= O(\epsilon^2).
\end{aligned}
$$

This imaginary system describes the evolution of the imaginary part of fields. These equations provide a way to carry information on the real equations. Indeed, depending on the choice and the nature of imaginary variables and parameters imposed initially and at the boundary computational domain, the solution of the imaginary equations provide different solution that can be related to various investigations. In the following, it is demonstrated that CVM allow:

- to compute the sensibility to a chosen parameter,
- to track an acoustic wave through the flow.

2.1 Sensibility analysis

The sensibility analysis of a specified parameter onto the variable consists in analyzing the impact of changing this parameter onto the solution of previous equations, for each time step. Such sensibility analysis can be performed by computing the partial derivative of each variable (here $\phi_a = p_a, \rho_a, \mathbf{v}_a, s_a$) of this parameter called α:

$$\phi_a^\alpha = \frac{\partial \phi_a}{\partial \alpha}. \quad (6)$$

α can be a *numerical* parameter (related for instance to the computational domain (mesh) or to the numerical schemes, ...) or a *physical* parameter (such as the size of an obstacle, the velocity of flow, the frequency of a wave, ...).
ϕ_a^α can also be determined by solving the so-called equations of sensibility. These are defined as the partial derivatives of the initial equations (system 1) relative to α, as follows:

$$
\begin{aligned}
\frac{\partial}{\partial\alpha}\left(\frac{\partial \rho_a}{\partial t} + \nabla.\left(\rho_a\mathbf{v}_a + \rho_0\mathbf{v}_a + \rho_a\mathbf{v}_0\right)\right) &= \frac{\partial M}{\partial \alpha},\\
\frac{\partial}{\partial\alpha}\left(\rho_0(\frac{\partial \mathbf{v}_a}{\partial t} + \mathbf{v}_0\nabla.\mathbf{v}_a + \mathbf{v}_a\nabla.\mathbf{v}_0) + \rho_a\mathbf{v}_0\nabla.\mathbf{v}_0 + \nabla p_a\right) &= \frac{\partial \mathbf{F}}{\partial \alpha},\\
\frac{\partial}{\partial\alpha}\left(\frac{\partial s_a}{\partial t} + \mathbf{v}_0\nabla s_a + \mathbf{v}_a\nabla s_0\right) &= 0,\\
\frac{\partial}{\partial\alpha}\left(p_a - c_0^2\rho_a - \Upsilon s_a\right) &= 0. \quad (7)
\end{aligned}
$$

Depending on the parameter selected for this analysis, the sensibility equations are different. But for each parameter, it is possible to match the sensibility equations and the imaginary equations. Let's take for instance the sensibility with respect to the celerity $\mathbf{c}_0$. Sensibility equations are:

$$
\begin{aligned}
\frac{\partial \rho_a^{c_0}}{\partial t} + \nabla.\left(\rho_a^{c_0}\mathbf{v}_a + \rho_0^{c_0}\mathbf{v}_a + \rho_a^{c_0}\mathbf{v}_0\right) &+ \quad (8)\\
\nabla.\left(\rho_a\mathbf{v}_a^{c_0} + \rho_0\mathbf{v}_a^{c_0} + \rho_a\mathbf{v}_0^{c_0}\right) &= M^{c_0},\\
\rho_0\left(\frac{\partial \mathbf{v}_a^{c_0}}{\partial t} + \mathbf{v}_0\nabla.\mathbf{v}_a^{c_0} + \mathbf{v}_0^{c_0}\nabla.\mathbf{v}_a + \mathbf{v}_a\nabla\mathbf{v}_0^{c_0} + \mathbf{v}_{a^{c_0}}\nabla\mathbf{v}_0\right) &+ \\
\rho_a^{c_0}\mathbf{v}_0\nabla.\mathbf{v}_0 + \rho_0^{c_0}\left(\frac{\partial \mathbf{v}_a}{\partial t} + \mathbf{v}_0\nabla.\mathbf{v}_a + \mathbf{v}_a\nabla\mathbf{v}_0\right) &+ \\
\rho_a\left(\mathbf{v}_0\nabla.\mathbf{v}_0^{c_0} + \mathbf{v}_0^{c_0}\nabla.\mathbf{v}_0\right) + \nabla p_a^{c_0} &= \mathbf{F}^{c_0},\\
\frac{\partial s_a^{c_0}}{\partial t} + \mathbf{v}_0\nabla s_a^{c_0} + \mathbf{v}_0^{c_0}\nabla s_a + \mathbf{v}_a^{c_0}\nabla s_0 + \mathbf{v}_a\nabla s_0^{c_0} &= 0,\\
p_a^{c_0} - 2c_0\rho_a - c_0^2\rho_a^{c_0} - \Upsilon^{c_0}s_a - s_a^{c_0}\Upsilon &= 0.
\end{aligned}
$$

One can note similarities between the sensibility equations (system 8) and the imaginary equations (system 11). Imaginary variables are governed by the same equations that sensibility variables. Then to compute the sensitivity to c_0, one has to set $c_{0I} = 1$, or in other words, to perturbe c_0 with a small imaginary part, and only this parameter.
Generally, imaginary equations match sensibility equations by perturbing the parameter by $i\epsilon$. This result can be proved with a Taylor expansion. For instance, for the variable p_a:

$$\bar{p}_a(x,y,t;\alpha + i\epsilon) = p_a(x,y,t;\alpha) + i\epsilon\frac{\partial p_a(x,y,t;\alpha)}{\partial\alpha} + O(\epsilon^2). \quad (9)$$

Thus, whatever the parameter disturbed, the imaginary part of each variable contains, for all time steps, the sensibility of the variable with respect to this parameter. In this case, the imaginary part is the solution of the sensibility system.

2.2 Wave tracking method

The wave tracking method consists in firstly discriminating an acoustic event from the flow field. Then, this method aims at following the propagation of this wave in the presence of a flow.

Based on CVM application and the resulting equations associated with the imaginary part of the complex system (system 11), it is quite interesting to note that when imposing the following conditions:

$$p_{0I} = \rho_{0I} = \mathbf{v}_{0I} = s_{0I} = c_{0I} = \Upsilon_I = 0, \quad (10)$$

the equations for imaginary fields are the same as those describing the propagation of the real acoustical disturbances (system 1):

$$
\begin{aligned}
\frac{\partial \rho_{aI}}{\partial t} + \nabla.\left(\rho_{aI}\mathbf{v}_a + \rho_{aI}\mathbf{v}_0 + \rho_a\mathbf{v}_{aI} + \rho_0\mathbf{v}_{aI}\right) &= M_I,\\
\rho_0\left(\frac{\partial \mathbf{v}_{aI}}{\partial t} + \mathbf{v}_0\nabla.\mathbf{v}_{aI} + \mathbf{v}_{aI}\nabla\mathbf{v}_0\right) &+ \\
\rho_{aI}\mathbf{v}_0\nabla.\mathbf{v}_0 + \nabla p_{aI} &= \mathbf{F}_I + O(\epsilon^2),\\
\frac{\partial s_{aI}}{\partial t} + \mathbf{v}_0\nabla s_{aI} + \mathbf{v}_{aI}\nabla s_0 &= 0, \quad (11)\\
p_{aI} - c_0^2\rho_{aI} - s_{aI}\Upsilon &= O(\epsilon^2).
\end{aligned}
$$

In this case, the governing equations of the imaginary part mimic exactly the equations of the real part. Hence, by choosing the same initial and boundary conditions and by imposing an imaginary part of the acoustic pressure field that corresponds to its real part, the solutions of both problems (real and imaginary) are exactly the same. It enables the tracking of the part of the field imposed in the imaginary part.

3 Numerical applications with CVM

3.1 Numerical resolution

The code, based on the pseudo-characteristic formulation[10], has already been used and presented in previous studies[4, 5]. It solves Navier-Stokes equations, without linearization and handles Complex Variable Methods utilization. Depending on the kind of analysis (sensibility, tracking), an imaginary part is added to a parameter or a variable. In every case, the parameter ϵ of the imaginary part has to be small enough to ensure that the terms of magnitude ϵ^2 are negligible. It can be chosen as small as desired, within the limits of computer precision. In the present study, $\epsilon = 10^{-8}$.

3.2 Application cases

As stated in the introduction part, the purpose of this work is to analyze acoustic waves propagating in a flow. For such a purpose a simplified two-dimensional $(\mathbf{e}_x, \mathbf{e}_y)$ aeroacoustic configuration is retained. The geometry of the problem is depicted in figure 1. It's an open area with a Rankine Vortex, and two acoustic sources.

The Rankine vortex model is a circular flow in which an inner circular region about the origin is in solid rotation, while the outer region is free of vorticity, the speed being inversely proportional to the distance from the origin. The velocity $\mathbf{v}(r, \theta)$ of a Rankine vortex with circulation Γ_0 and radius r_0 is defined, in polar coordinates (r, θ), as:

$$\mathbf{v}(r \leq r_0, \theta) = \frac{\Gamma_0 r}{2\pi r_0^2}\mathbf{e}_\theta, \quad (12)$$

$$\mathbf{v}(r > r_0, \theta) = \frac{\Gamma_0}{2\pi r}\mathbf{e}_\theta. \quad (13)$$

In addition, two acoustic sources $M(x, y, t)$ and $\mathbf{F}(x, y, t)$ are positioned close to the vortex, at a distance of about 3 times the characteristic size of the vortex. $M(x, y, t)$ corresponds to a mass injection (an acoustic monopole) of amplitude M_0, with a Gaussian profile of standard deviation σ_m in space and of frequency f. $\mathbf{F}(x, y, t)$ corresponds to a force injection (an acoustic dipole) of amplitude $\mathbf{F}_0$, with a Gaussian profile of standard deviation σ_f in space and of frequency f:

$$M(x, y, t) = M_0 e^{-\frac{(x-x_m)^2+(y-y_m)^2}{2\sigma_m^2}} \sin(2\pi f t) \quad (14)$$

$$\mathbf{F}(x, y, t) = \mathbf{F}_0 e^{-\frac{(x-x_f)^2+(y-y_f)^2}{2\sigma_f^2}} \sin(2\pi f t) \quad (15)$$

The physical dimensions of the domain are $(L_x, L_y) = (20\,m, 20\,m)$ that correspond to a uniform cartesian mesh of $(N_x \times N_y) = (200 \times 200)$ points. The time step is $dt = 1.2 \times 10^{-4}\ s$. The computational domain is of size $L_x \times Ł_y$.

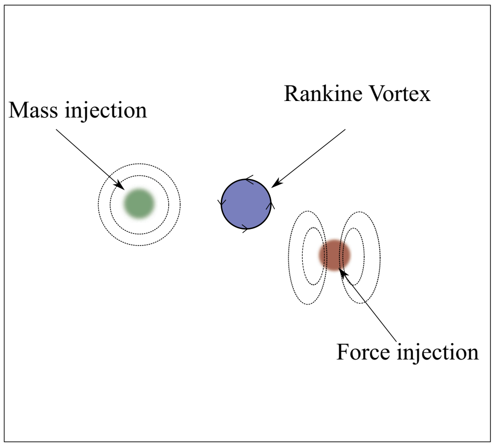

Figure 1: Schematization of the cases used to illustrate the method of complex variables. The aeroacoustic field is composed by two acoustic sources, and a Rankine vortex.

Acoustic sources have a standard deviation $\sigma_f = \sigma_m = 0.3\,m$. The acoustic frequency is $f = 200\,Hz$. The acoustic monopole is located at $(x_m, y_m) = (-3, 0)$. The acoustic dipole is located at $(x_f, y_f) = (2, -1)$. Amplitudes are $M_0 = 0.55\,kg.m^{-3}.s^{-1}$ and $\mathbf{F}_0 = 160.\mathbf{e}_x\,kg.m^{-2}.s^{-2}$. The Rankine vortex has a circulation $\Gamma_0 = 300\,m^2/s$, a radius $r_0 = 1\,m$, and is positioned at $(0, 0)$.

The figure 2 shows the fluctuating pressure. The scale has been saturated in order to observe the acoustic pressure field. One observe that the monopole is fairly visible, unlike the dipole. In this case, it is very tricky to extract the acoustic source associated with the mass injection. A Fourier transformation (temporal or spatial) would be useless because of the proximity of frequencies and wave-vectors. Nevertheless, Complex Variable Methods allow to overcome this issue.

In the following, imaginary parts divided by ϵ are shown.

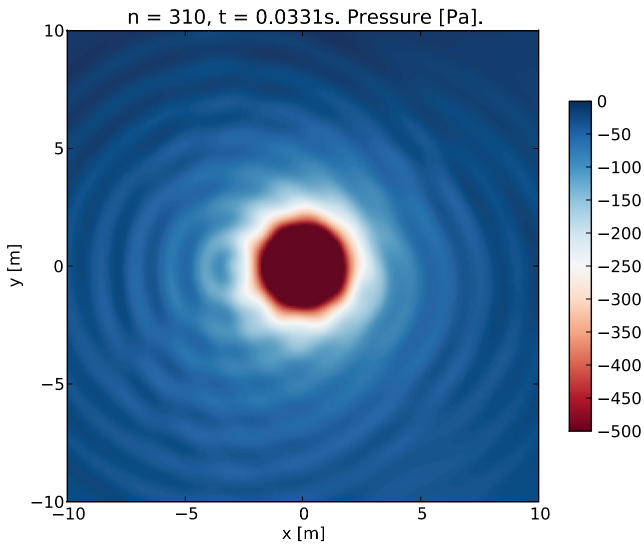

Figure 2: Total fluctuating pressure field.

3.3 Sensibility analysis

The sensibility analysis of the acoustic fields with respect to a geometrical parameter characterizing the acoustic source is presented. To illustrate the method, the sensibility to the standard deviation σ_m, a parameter characterizing the width of the acoustic monopole, is conducted. According to the discussion of section III, to analyze the sensibility to σ_m, one has to impose a small imaginary part to this parameter:

$$\bar{\sigma_m} = \sigma_m + i\epsilon. \quad (16)$$

From a numerical point of view, this can be done by imposing the following source term in the code:

$$\bar{M}(x, y, t) = e^{-\frac{(x-x_m)^2+(y-y_m)^2}{2(\sigma_m+i\epsilon)^2}} \sin(2\pi f t) \quad (17)$$

Results are presented for two different time steps in figures 3 and 4. At the time step $n = 1$ (figure 3), the sensibility of the source to σ_m is highest at the edges of the wave. Indeed, a variation of the size of the source has an obvious impact on the border of this one. Figure 4 shows that it is possible to extract directly from the code the sensibility to σ_m for each spatial point, and for each time step.

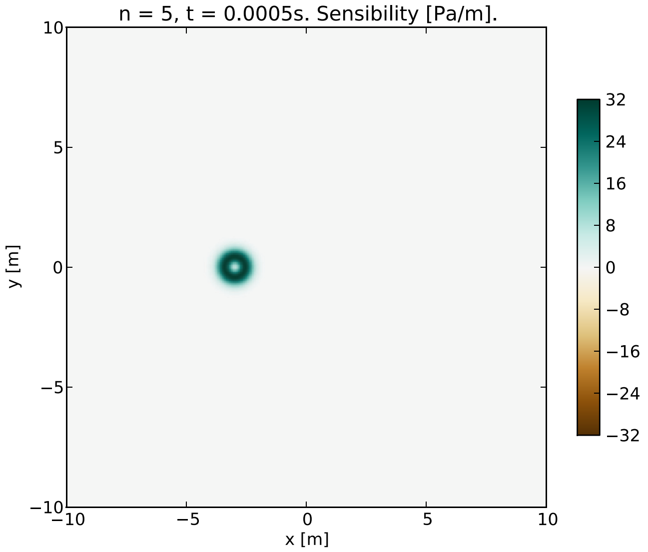

Figure 3: Imaginary part of pressure in a sensibility calculation configuration after 5 time steps.

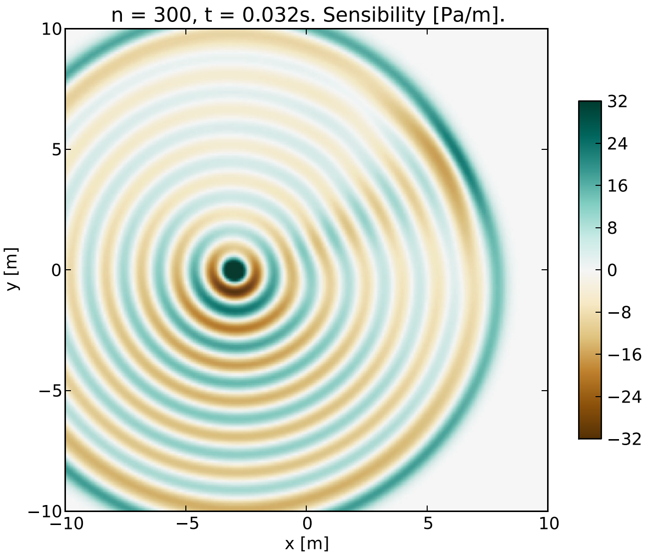

Figure 4: Imaginary part of pressure in a sensibility calculation configuration after 300 time steps.

3.4 Wave Tracking Method

In this part, the effectiveness of CVM in tracking selected waves is presented. From the theoretical part, it appears that wave tracking method allows to follow a part of the field. To do that, one has to tag the selected acoustic event by imposing a small imaginary part to it. From a numerical point of view, the imaginary part of the acoustic source is imposed similar to the real part, in accordance with theoretical background. For example, to track the acoustic dipole that doesn't clearly appears in the total pressure field (fig. 2), one has to impose:

$$\bar{\mathbf{F}}(x, y, t) = \mathbf{F}_0 e^{-\frac{(x-x_f)^2+(y-y_f)^2}{2\sigma_f^2}} \sin(2\pi f t)(1 + i\epsilon). \quad (18)$$

As shown previously, the pressure added to the imaginary part is solution of equations governing the evolution of the associated acoustic pressure in the total field. One can see in the figure 5 the acoustic dipole after 330 time steps, undergoing the effect of the flow. To measure the accuracy of the tracking method, a second simulation without the acoustic dipole has been conducted. This simulation allows to extract the acoustic dipole with a difference between the two pressure fields of both simulations. This reference is compared to the acoustic dipole obtained with the tracking method. The relative error is shown for the time step $n = 330$. It may be noted that locations where the error is the largest correspond to locations where the pressure is of greater amplitude. One obtains that the tracked acoustic wave is equal to the corresponding wave in the total field, with a relative error inferior than 0.07%.

Thereby, it is possible to extract very neatly and very simply any acoustic source, just by adding to it a small imaginary part, even in the presence of a steady flow.

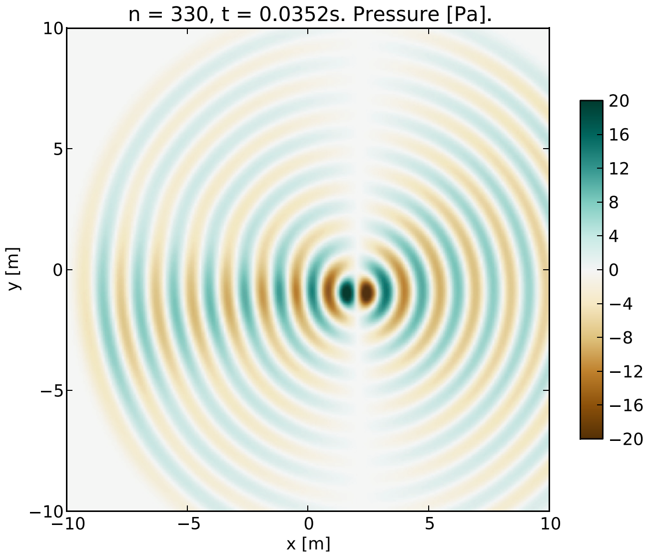

Figure 5: Imaginary part of pressure in a wave tracking configuration after 330 time steps.

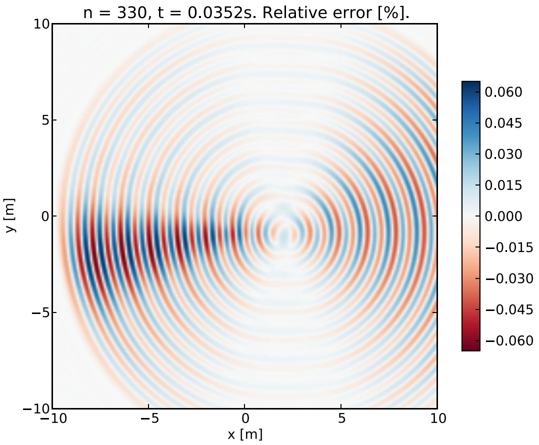

Figure 6: Relative error of the acoustic dipole extracted with CVM.

4 Conclusions

Sensibility analysis by Complex Variable Methods CVM) appears as a tool that can carry accurate informations about the source. In addition, CVM provide a way to track an acoustic wave, which enables to see the evolution of a selected source and understand how it evolves in a steady flow. These analyses are running during the simulation. No post-processing is required, and it is not necessary to save all steps of the initial simulation.

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