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Omnidirectional graded index sound absorber

O. Umnova and B. Zajamsek

University of Salford, The Crescent, Salford, m5 4wt Greater Manchester, UK
o.umnova@salford.ac.uk

It has been shown recently that a circular cylindrical layer with dielectric constant, altering as $1/r^n$ with radius r can capture incident light irrespective of the incidence angle and direct it towards the absorbing core hence achieving the "black hole" effect. If effective and compact acoustical analogues of such structures are developed, they will act as omnidirectional broadband absorbers of sound. Following the approach previously used to design flat gradient acoustic lens and electromagnetic black holes, a graded index metamaterial layer is designed using an array of small scatterers with their concentration varying along the structure radius. In an array of small rigid circular cylinders the effective density (which is the acoustical analogue of the dielectric constant) varies with filling fraction *ff* as $\rho_{eff} = \rho_0(1+ff)/(1-ff)$. This means that by varying the filling fraction the desired dependence of ρ_{eff} on r can be achieved. A porous material has been used as an absorbing core. The device is shown to provide a nearly total absorption of acoustic waves with wavelengths smaller than the structure diameter.

1 Introduction

An omnidirectional absorber of electromagmetic (EM) waves has received a significant amount of attention recently. It was shown that nearly total angle independent absorption of incident waves can be achieved using a cylindrical or a spherical device comprised of an absorbing core surrounded by a layer with dielectric constant varying with the distance r to the device centre [1]. In a layer that captures incident radiation irrespective of the incidence angle and guides it towards the absorbing core the dielectric constant should vary as $1/r^n$, where n is a positive integer bigger or equal than 2. At the boundary between this layer and the absorbing core a nearly perfect impedance matching is achieved as well as at the air/ layer interface, so the reflections are minimized. It has been shown, that for practical applications, the dielectric constant profile in the matching layer can be reasonably well approximated by several layers with varying dielectric constants [2]. The EM "black hole" device is effective for wavelengths that are smaller than its radius.

This work is an attempt to design an acoustic analogue of omnidirectional light absorber that works for airborne sound. Similar to EM case, a multilayered structure is used to approximate the matching layer profile. This has been done using several rows of small cylinders with filling fraction increasing towards the device centre. A granular porous material is used as an absorbing core.

There are notable differences between the optical and the acoustical cases. Firstly, radial variations of two material parameters in the matching layer should be accounted for. Two types of optical "black holes" have been reported – for TE and TM polarized light [2]. However in both cases it was possible to vary dielectric constant in an essentially non magnetic medium. It has been long established that acoustic analogues of dielectric constant and magnetic permeability are effective material density and bulk modulus. For a simplest effective medium, such as regular arrangement of subwavelength in size rigid cylinders in air, both effective density and bulk modulus have strong dependence on the filling fraction. This means that if this medium is used for the matching layer, then radial variations of its effective density are inevitably accompanied by corresponding variations of the effective bulk modulus. Secondly, if porous material is used as an absorbing core, the frequency dependence of its characteristic impedance and wavenumber should be accounted for in the design.

2 The Model

2.1 Scattering problem

Consider a cylindrical layer with outer radius R and inner radius R_c. and with radially varying effective density

$$\rho(r) = \rho_0 \alpha(r), \tag{1}$$

and effective bulk modulus

$$K(r) = \rho_0 c^2 \kappa(r). \tag{2}$$

Here ρ_0 is equilibrium density of air, c is speed of sound, $\alpha(r)$ and $\kappa(r)$ are normalized effective properties which will be considered in the following. The wavenumber $k(r)$ and the characteristic impedance $Z(r)$ of the layer vary with r according to

$$k(r) = k_0\sqrt{\alpha(r)/\kappa(r)},\ Z(r) = Z_0\sqrt{\alpha(r)\kappa(r)} \tag{3}$$

where $k_0 = \omega/c$, $Z_0 = \rho_0 c$ and ω is angular frequency of sound. Pressure inside the layer $R_c < r < R$ can be represented as a Fourier series

$$p_L = \sum_{m=-\infty}^{+\infty} p_m(r) e^{im\theta} \tag{4}$$

where r and $\theta$are cylindrical coordinates and $\boldsymbol{p_m(r)}$ is the solution of the following equation

$$(r^2 k(r)^2 - m^2)p_m + r\left(1 - \frac{r\alpha(r)'}{\alpha(r)}\right)p'_m + r^2 p''_m = 0. \tag{5}$$

Here $'$ stands for differentiation with respect to r. When $k(r) = k_0\sqrt{\alpha(r)}$, i.e when effective bulk modulus does not vary with radius and $\kappa(r) = 1$, equation (5) reduces to wave equation used in the design of the EM black hole for TM polarization [2]. In this case, the simplest capturing profile $k(r) \sim 1/r$ results in $\alpha(r) \sim 1/r^2$and equation (5) can be solved analytically. However, when variations of the effective bulk modulus are not negligible, the analytical solution does not seem possible. In this case if $k(r)\sim 1/r$ then dependence $\alpha(r)\sim 1/r^2$ is not achieved as demonstrated below.

Consider a regular array of infinitely long rigid cylinders with radius *a* and varying filling fraction ff*(r)*. For a low frequency wave (k_0a<<1) propagating in the plane perpendicular to cylinder axes this array behaves as a medium with the following effective properties [3]:

$$\alpha(r) = \frac{1+ff(r)}{1-ff(r)} \qquad \kappa(r) = \frac{1}{1-ff(r)} \tag{6}$$

According to this and equation (3) the wave number and characteristic impedance of this medium vary as

$$k(r) = k_0\sqrt{1+ff(r)}, \qquad Z(r) = Z_0\frac{\sqrt{1+ff(r)}}{1-ff(r)}. \tag{7}$$

To achieve $k(r)\sim 1/r$ so that

$$k(r)r = k_0 R, \tag{8}$$

the filling fraction should increase towards the centre of the cylinder as

$$ff(r) = \left(\frac{R}{r}\right)^2 - 1, \tag{9}$$

which leads to the following radial dependence of the normalized effective density

$$\alpha(r) = \frac{1}{2\left(\frac{r}{R}\right)^2 - 1}. \tag{10}$$

This obviously differs from $\alpha(r)\sim 1/r^2$ dependence which would have been achieved in case of constant effective bulk modulus. Substituting (8) and (10) in (5), the following equation for p_m is derived:

$$\left(R^2{k_0}^2 - m^2\right)p_m + r\frac{6r^2-R^2}{2r^2-R^2}p_m' + r^2 p_m'' = 0, \tag{11}$$

which will be solved numerically.

Pressure inside the absorbing core, $r < R_c$, is described as

$$p_i = \sum_{m=-\infty}^{+\infty} A_m J_m(k_i r)e^{im(\theta+\pi/2)}, \tag{12}$$

here k_i is wavenumber of the absorbing material and J_m is Bessel function of the first kind of order *m*.

Outside the cylinder, $r > R$, the field is decomposed into the incident (plane) wave and a scattered wave

$$p_o = \sum_{m=-\infty}^{+\infty} \left(J_m(k_0 r) + B_m H_m(k_0 r)\right)e^{im(\theta+\pi/2)}, \tag{13}$$

where H_m is Hankel function of the first kind of order *m*.

Boundary conditions of pressure and radial velocity continuity are satisfied at $r=R_c$ and $r=R$. This results in the following set of equations:

$$A_m J_m(k_i R_c) = p_m(R_c), \tag{14,a}$$

$$\frac{A_m}{\alpha_i}J_m{}'(k_i R_c) = \frac{1}{\alpha(R_c)}p_m'(R_c), \tag{14.b}$$

$$J_m(k_0 R) + B_m H_m(k_0 R) = p_m(R), \tag{14,c}$$

$$J_m'(k_0 R) + B_m H_m'(k_0 R) = p_m'(R). \tag{14,d}$$

The fact that $\alpha(R) = 1$ was taken into account here.

Then scattering coefficients B_m can be expressed as

$$B_m = \left(\frac{-J_m'(k_0R) + \xi_m J_m(k_0R)}{H_m'(k_0R) - \xi_m H_m(k_0R)}\right), \tag{15}$$

where

$$\xi_m = \frac{p_m'(R)}{p_m(R)}$$

can be found solving equation (11) numerically with boundary condition at $r = R_c$:

$$\frac{p_m'(R_c)}{p_m(R_c)} = \frac{\alpha(R_c)}{\alpha_i}\frac{J_m{}'(k_i R_c)}{J_m(k_i R_c)}.$$

In EM device the core radius R_c was chosen to match the real part of the absorbing core dielectric constant and the effective dielectric constant of the matching layer [1]-[2]. For the acoustic analogue this approach results in matching effective density: $\alpha(R_c) = Re(\alpha_i)$. Using (10) the following expression for the core radius is derived:

$$R_c = \frac{R}{\sqrt{2}}\sqrt{\frac{Re(\alpha_i)+1}{Re(\alpha_i)}}. \tag{16}$$

The maximum filling fraction of small cylinders in matching layer is

$$ff(R_c) = \frac{Re(\alpha_i) - 1}{Re(\alpha_i) + 1}. \quad (17)$$

This obviously has to be lower than close packing filling fraction which limits the choice of materials for the absorbing core.

Two alternative approaches to the design are based on matching real part or absolute value of normalized impedance $z_i = k_0 \frac{\alpha_i}{k_i}$ of the core and characteristic impedance $Z(R_c)$. This results in the following core radius

$$R_c = \frac{R}{\sqrt{2}} \frac{\sqrt{8q^2}}{\sqrt{8q^2+1} - 1} \quad (18)$$

and maximum filling fraction of cylinders in a matching layer

$$ff(R_c) = 1 - \frac{\sqrt{8q^2+1} - 1}{2q^2}, \quad (19)$$

where $q = Re(z_i)$ or $q = |z_i|$. In this work the calculations have been performed assuming that $Z(R_c) = |z_i|$. It follows from (16) and (18), that absorbing core radius is larger then $R/\sqrt{2}$ if effective medium of small cylindrical scatterers is used for matching layer.

The omnidirectional absorber considered here is intended for frequencies between 500Hz and 1500Hz (wavelengths 23-69cm). The structure with radius significantly exceeding the wavelength would be too big to be practical. The outer radius R=70cm was chosen. This means that the absorber is not operating in the regime where it is supposed to be most effective [1]-[2].

2.2 Properties of porous absorbing core

A packing of perfectly spherical particles with radius of 2mm is chosen for the absorbing core. Acoustical parameters of this material are given in [4]. The calculated frequency dependence of the normalized characteristic impedance and wavenumber is shown in Figure 1 (a,b). In the frequency range 300-1600Hz, the real part of normalized impedance varies between 4.06 to 3.62, while its imaginary part decreases from 0.68 to 0.33. Real and imaginary parts of the normalized wavenumber are confined between 1.82-1.52 and 0.44-0.22 respectively.
To achieve perfect impedance matching between the matching layer and the absorbing core, different values of R_c and ff(R_c) would be necessary at different frequencies. This however contradicts the idea of a broadband absorber. For this reason these quantities were calculated to achieve perfect matching for the lowest operating frequency, i.e. 500Hz. This results in R_c=54cm, ff(R_c)=0.67.

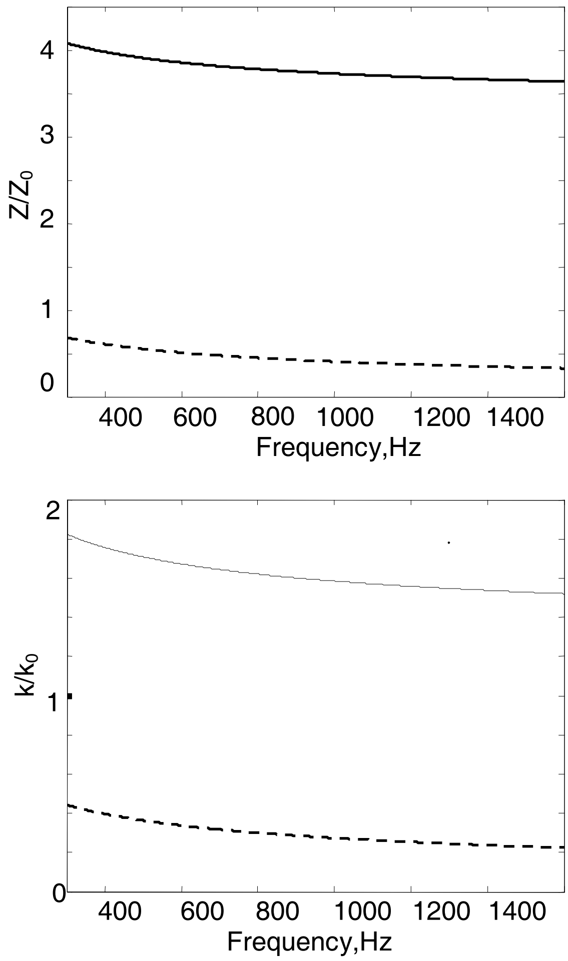

Figure 1. Frequency dependence of the normalized characteristic impedance (a) and wavenumber (b). Packing of identical spheres, radius 1.89mm. Solid lines – real parts, dashed lines – imaginary parts.

2.3 Approximation of the matching layer using multiple rows of cylinders

Continuous radial variations of the effective density and bulk modulus in the matching layer are approximated using ten rows of rods with filling fractions increasing towards the cylinder centre. The rods radii a are chosen between 3mm and 6mm. The properties of the layers are summarised in Table 1. To justify this choice the following calculations have been performed. First of all, in the frequency range of interest maximum $k_0 a$ value achieved is less than 0.15. This means that rods are subwavelength in size. Using formalism developed in [5], reflection and transmission coefficients of a single row of small equally spaced rigid cylinders can be calculated and then compared with those for a layer of material with properties defined by (6). Layer thickness is assumed to be equal to the lattice constant: $L = a\sqrt{\frac{\pi}{ff}}$. Reflection coefficient is defined as the ratio of the reflected wave pressure to that of the incident plane wave at the front surface of the layer. Transmission coefficient is defined as the ratio of the transmitted wave pressure to that of the incident wave at the back of the layer. While the agreement is very good at relatively low filling fraction of cylinders, it deteriorates when cylinders are spaced closely. However as shown in Figure 2 even for the last layer (Table 1) with filling fraction of 0.67 a satisfactory agreement between two sets of results is still achieved.

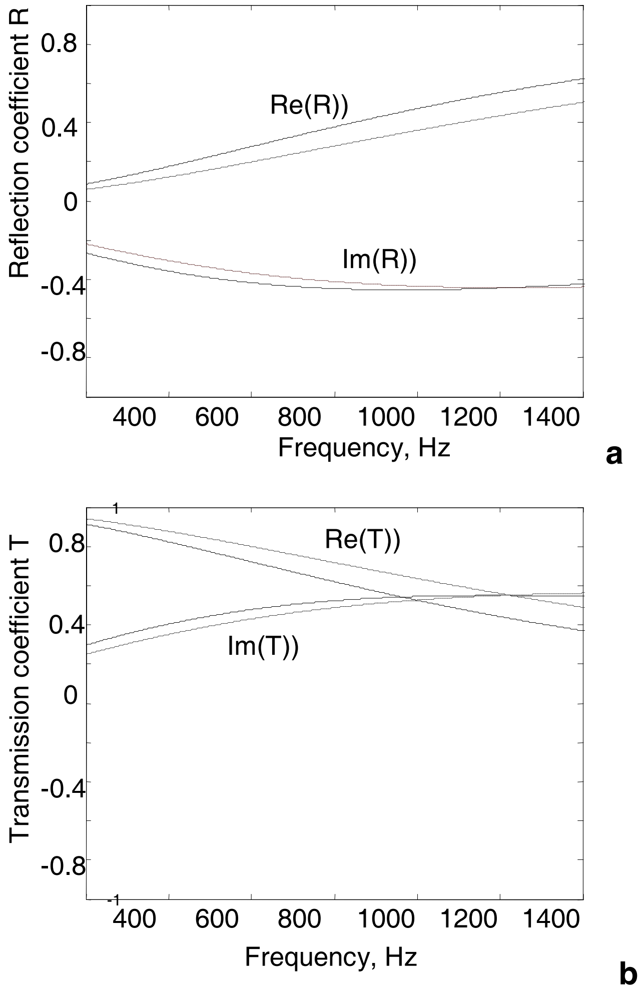

Figure 2. Frequency dependence of reflection and transmission coefficients. Solid lines – calculations for a row of cylinders [5], dashed line –calculations for a layer of material with thickness L and properties defined by (6). Cylinder radius is 6mm, filling fraction is 0.67.

Table 1: Approximation of matching layer by rows of cylinders

Layer number	Cylinder radius,cm	Filling fraction	Lattice constant,cm
1	0.3	0.05	2.3
2	0.4	0.12	2.1
3	0.5	0.19	2.0
4	0.5	0.26	1.7
5	0.5	0.32	1.6
6	0.5	0.39	1.4
7	0.5	0.45	1.3
8	0.6	0.55	1.4
9	0.9	0.60	1.4
10	0.6	0.67	1.3

Figure 3 shows a continuous filling fraction profile of the matching layer defined by (9) and its approximation with ten rows of small cylinders with properties defined in Table 1.

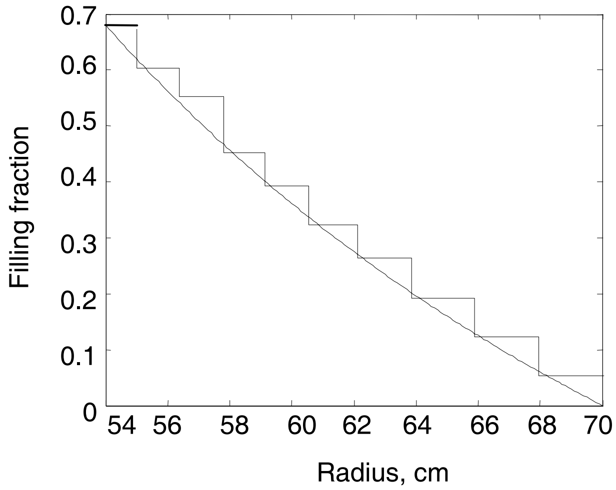

Figure 3. Continuous filling fraction profile defined by (9) and its approximation with ten layers (properties as in Table 1).

Losses in the matching layer can be neglected if viscous boundary layer $\delta(\omega)=\sqrt{\frac{2\eta}{\omega\rho_0}}$ is less than minimum distance between the rods $d = a\left(\sqrt{\frac{\pi}{ff}}-2\right)$. Maximum value of $\delta(\omega) = 0.18mm$ is achieved at 500Hz. Minimum value of d is $1mm$ (layer number 10). This suggests that although viscosity effects can be neglected in most layers, at low frequencies they can be noticeable in the last layer. The model presented here does not account for viscosity so its predictions are less accurate at low frequencies.

3 Results

Polar plots showing amplitude of the scattered wave normalised to that of the incident wave for three frequencies (500Hz, 100Hz and 1500Hz) are shown in Figure 4 (a-c). The plane wave is incident from the left. The results for a continuous matching layer profile (solid black lines) are very close to that for a multilayered structure (dashed black lines). To emphasize the effect of the matching layer, results for omnidirectional absorber are compared with those for a porous cylinder made of absorbing core material (radius 70cm).

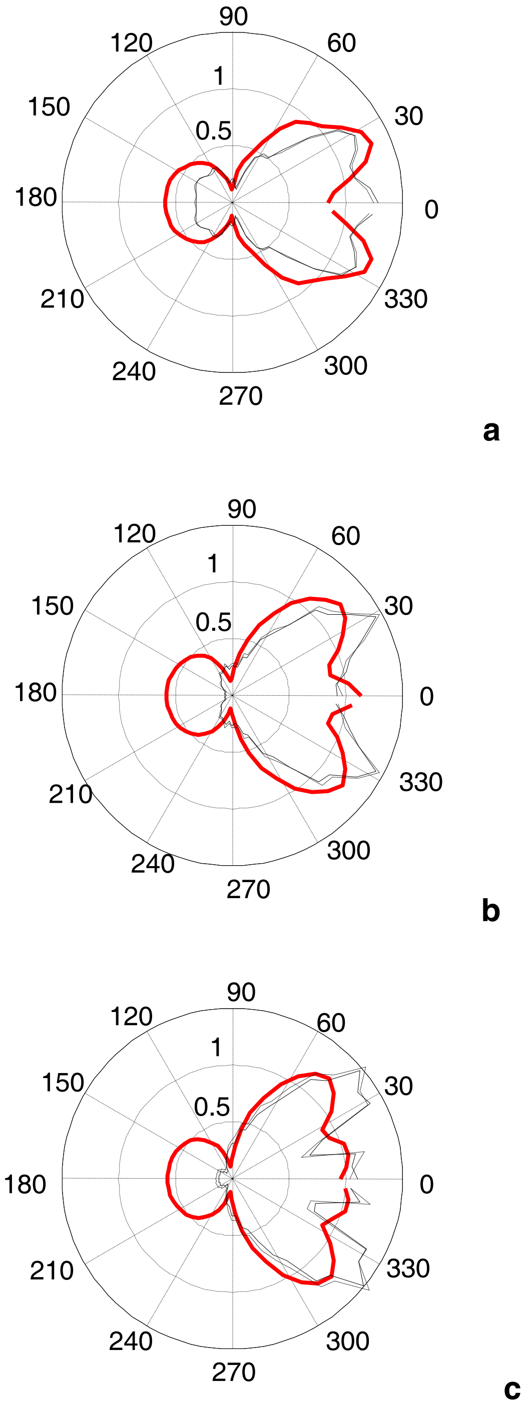

Figure 4. Normalised amplitude of the scattered field on the absorber surface versus angle, 500Hz (a), 1000Hz (b), 1500 Hz (c). Plane wave incident from the left (180^o). Red line – porous absorber without matching layer, black solid line – omnidirectional absorber with continuous matching layer, black dashed lines – omnidirectional absorber with multilayered matching layer.

The amplitude of the wave reflected from porous cylinder towards the source (180^o) exceeds 0.5 for all three frequencies. With matching layer this is decreased to 0.32 at 500Hz, 0.07 at 1000Hz and 0.11 at 1500Hz.

Distribution of the total field is shown in Figure 5 for porous cylinder without matching layer (a) and for omnidirectional absorbing device (b). The frequency is 1000Hz. A strong interference between the incident and the reflected waves in front of the porous cylinder can be clearly observed. In case of omnidirectional absorber the interference is weak due to strongly reduced reflections from its surface (see Figure 4,b).

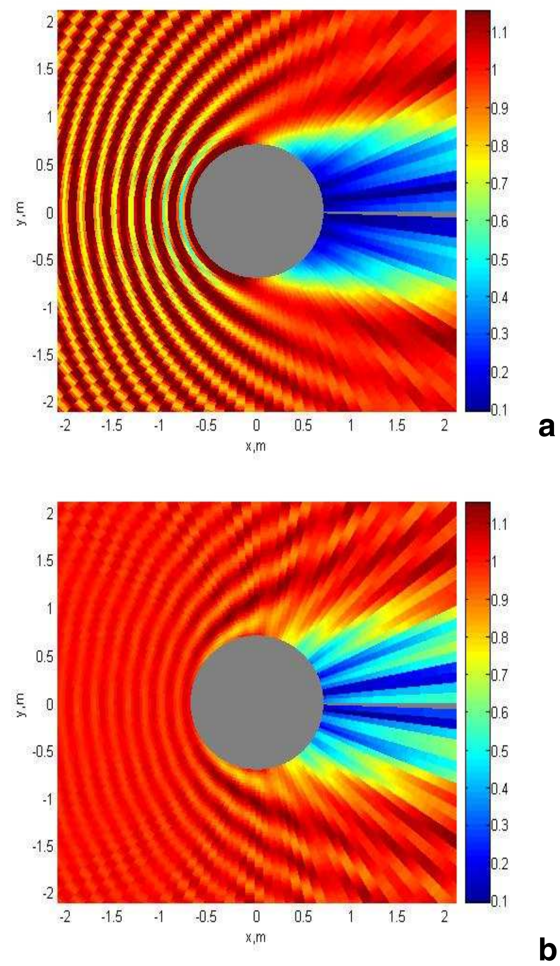

Figure 5. Normalised amplitude of the total field around porous absorber (a) and omnidirectional absorber with continuous matching layer (b). Sound frequency is 1000Hz. Plane wave is incident from the left (180^o).

Efficiency of the omnidirectional absorber can be calculated similar to [2] and is 85.1% at 500Hz, 94.8% at 1000Hz and 95.2% at 1500Hz.

4 Conclusions

A design for omnidirectional absorber for airborne sound is suggested. Similar to its EM analogue, the cylindrical device consists of the absorbing core surrounded by an impedance matching layer. The model is developed which accounts for radial variations of both effective density and effective bulk modulus in the matching layer. For practicality the required matching layer profile is approximated by ten rows of small cylinders with filling fraction increasing towards the centre. A granular porous material has been used for the absorbing core. It has been demonstrated that the structure efficient in the reasonably wide range of frequencies can be designed even when the core properties are frequency dependent. The device size could be the main obstacle for its practical applications. However, it is shown that even when sound wavelength is comparable to the device radius it is still more efficient than a conventional porous absorber of the same size. The absorption efficiency between 85.1% and 95.2% in the frequency range 500-1500Hz can be achieved using 70cm radius device.

References

[1] E.E.Narimanov, A.V.Kildishev "Optical black hole: broadband omnidirectional light absorber", *Appl. Phys. Lett.* 95, 041106 (2009)

[2] A.V.Kildishev L.J.Prokopeva, E.E.Narimanov "Cylinder light concentrator and absorber: theoretical description", *Optics Express* 18(16), 16646-16662 (2010)

[3] J.Mei, Z.Liu, W.Wen, P.Sheng "Effective mass density of fluid-solid composites" *Phys. Rev. Lett.* 96, 024301 (2006)

[4] O.Umnova, K.Attenborough, E.Standley, A.Cummings "Behaviour of rigid-porous layers at high levels of continuous acoustic excitations: theory and experiment", *J. Acoust. Soc. Am.* 114(3), 1346-1356, (2003)

[5] V.Twersky, "On scattering of waves by the infinite grating of circular cylinders", *IRE Trans. on Antennas and Propagation*, 10(6), 736-765 (1962)