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Hyperelastic cloaking theory: Transformation elasticity with prestressed solids

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Transformation elasticity, by analogy with transformation acoustics and optics, converts material domains without altering wave properties, thereby enabling cloaking and related effects. By noting the similarity between transformation elasticity and the theory of incremental motion superimposed on finite pre-strain it is shown that the constitutive parameters of transformation elasticity correspond to the density and moduli of small-on-large theory. We first consider antiplane wave cloaks, generated from neo-Hookean hyperelastic media before going on to consider the more general theory of elastodynamics. In the latter, the formal equivalence indicates that transformation elasticity can be achieved by selecting a particular finite (hyperelastic) strain energy function. The uniquely defined form for isotropic elasticity is semilinear strain energy. The associated elastic transformation is restricted by the requirement of statically equilibrated pre-stress. This constraint can be cast as tr$\mathbf{F}$ = constant, subject to symmetry constraints, and its consequences are explored both analytically and through numerical examples of cloaking of anti-plane and in-plane wave motion.

1 Introduction

The principle underlying cloaking theory is the transformation method whereby the material properties of the cloak are defined by a (singular) spatial transformation. For elastodynamics, Milton et al. [1] concluded that the transformed materials are described by the Willis model, involving coupling between stress and velocity, in addition to anisotropic inertia. For a restricted transformation, Brun et al. [2] found transformed material properties with isotropic inertia and elastic behavior of Cosserat type, i.e. with properties that are the same as those of "standard" linear elasticity except that the moduli do not satisfy the minor symmetry, i.e. $C^*_{jikl} \neq C^*_{ijkl}$.

The transformed elastodynamic constitutive parameters may be characterized through their dependence on (i) the transformation (mapping function) and (ii) the relation between the displacement fields in the two descriptions, represented by matrices: $\mathbf{F}$, the deformation gradient matrix, and $\mathbf{A}$, respectively. It was shown in [3] that requiring stress to be symmetric implies $\mathbf{A} = \mathbf{F}$ and that the material must be of Willis form, as in [1]. Setting $\mathbf{A} = \mathbf{I}$, on the other hand, results in Cosserat materials with non-symmetric stress but isotropic density, as found by Brun et al. [2].

In this paper we consider a class of materials displaying non-symmetric stress of the type necessary to achieve elastodynamic cloaking by taking advantage of the similarities between transformation elasticity and small-on-large motion in the presence of finite pre-strain [4]. Such an approach has already been shown to be successful; by using an incompressible neo-Hookean material with a radially symmetric cylindrical pre-strain, Parnell [5] showed that the resulting small-on-large equations are identically those required for cloaking of the horizontally polarized shear (SH or antiplane) wave motion. In [5] the pre-stress affected the entire elastic domain however and therefore its influence was felt by both the source and receiver. In this letter we show how this theory may be adapted in order to create a finite cloak by means of an axial stretch. We then go on to consider the general elastodynamic transformation problem, including but not limited to SH motion.

2 Finite cloaks for antiplane waves

As noted in [2], a special case for elastodynamics is the antiplane elastic wave problem, where cloaking can readily be achieved from a cylindrical region (using a cylindrical cloak) in two dimensions by virtue of the duality between antiplane waves and acoustics in this dimension. Consider an unbounded homogeneous elastic material with shear modulus μ_0 and density ρ_0 and introduce a Cartesian coordinate system (X, Y, Z) and cylindrical polar coordinate system (R, Θ, Z) with some common origin $\mathbf{O}$. Planar variables are related in the usual manner, $X = R\cos\Theta, Y = R\sin\Theta$. Suppose that there is a time-harmonic line source, polarized in the Z direction and located at (R_0, Θ_0), with circular frequency ω and amplitude C (which is a force per unit length in the Z direction). This generates antiplane elastic waves with the only non-zero displacement component in the Z direction of the form $\mathrm{U} = \Re[W(X, Y)\exp(-i\omega t)]$. The displacement W is governed by

$$\nabla_{\mathbf{X}} \cdot (\mu_0 \nabla_{\mathbf{X}} W) + \rho_0 \omega^2 W = C\delta(\mathbf{X} - \mathbf{X}_0), \tag{1}$$

where $\nabla_{\mathbf{X}}$ is the gradient operation in the "untransformed" frame, $\mathbf{X} = (X, Y)$ and $\mathbf{X}_0 = (X_0, Y_0)$.

The assumed mapping for a cloak for antiplane waves (cf. acoustics) expressed in plane cylindrical polar coordinates, takes the form

$$r = g(R), \qquad \theta = \Theta, \qquad z = Z, \quad \text{for } 0 \leq R \leq R_2, \tag{2}$$

and the identity mapping for all $R > R_2$ for some chosen monotonically increasing function $g(R)$ with $g(0) \equiv r_1 \in [0, R_2]$, $g(R_2) = R_2 \in \mathbb{R}$ such that $R_2 < R_0$, i.e. the line source remains outside the cloaking region. The cloaking region is thus defined by $r \in [r_1, r_2]$ where $r_2 = R_2$. We use upper and lower case variables for the untransformed and transformed problems respectively. Under this mapping the form of the governing equation (1) remains unchanged for $R = r > R_2$, whereas for $0 \leq R \leq R_2$, corresponding to the transformed domain $r_1 \leq r \leq R_2$, the transformed equation takes the form (in transformed cylindrical polar coordinates $r, \theta = \Theta$)

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\mu_r(r)\frac{\partial w}{\partial r}\right) + \frac{\mu_\theta(r)}{r^2}\frac{\partial^2 w}{\partial \theta^2} + d(r)\omega^2 w = 0 \tag{3}$$

where (see eqs. (26), (27) in [3])

$$\mu_r(r) = \frac{\mu_0^2}{\mu_\theta(r)} = \mu_0 \frac{R}{r}\frac{\mathrm{d}g}{\mathrm{d}R}, \quad d(r) = \rho_0 \frac{R}{r}\left(\frac{\mathrm{d}g}{\mathrm{d}R}\right)^{-1}. \tag{4}$$

Hence, *both* the shear modulus and density must be inhomogeneous and the shear modulus must be anisotropic. Material properties of this form cannot be constructed exactly since the shear modulus μ_θ becomes unbounded as $r \to r_1$ (the inner boundary of the cloak). In this limit the density behaves as $d = (pcr_1)^{-1}\rho_0 R^{2-p} + \ldots$ where p, $c > 0$ define the mapping in the vicinity of the inner boundary according to $r = r_1 + cR^p + \ldots$ as $R \to 0$. In practice of course approximations are required as described in e.g. [6, 7, 8]. Note that, as expected [3], the total mass is conserved since, regardless of the mapping, the integral of the density $d(r)$ over $r \in [r_1, r_2]$ is $\pi R_2^2 \rho_0$.

In [5] a new method to generate elastic cloaks was proposed which used the notion of nonlinear pre-stress. That this was possible was due to the fact that the antiplane wave field scattered from a cylindrical cavity *is invariant under pre-stress for an incompressible neo-Hookean material*. Scattering coefficients in the deformed configuration depend only on the *initial* cavity radius R_1 and therefore provided that this is small compared with the incident wavelength, scattering from the inflated cavity of radius r_1 will be negligible regardless of the relative size of r_1 and the incident wavelength. Therefore we can conclude that an object placed inside the inflated cavity region would be near-invisible (i.e. cloaked) upon choosing R_1 appropriately. In [5] the pre-stress affected the entire elastic domain ,i.e. its influence was felt by both the source and receiver. Let us now show how we may adapt this theory in order to create a finite cloak by means of an axial stretch. Full details are given in [9].

With reference to Fig. 1, let us consider an elastic material within which is located a cylindrical cavity of radius R_2. Let us assume that the density of this medium is ρ_0 and its axial shear modulus (corresponding to shearing on planes parallel to the axis of the cylindrical cavity) is μ_0. Additionally we take a cylindrical annulus of isotropic *incompressible* neo-Hookean material with associated shear modulus μ and density ρ and with inner and outer radii R_1 and R_2 respectively with $R_1 \ll R_2$. The exact nature of this latter relationship will be described shortly. We shall consider deformations of the cylindrical annulus in order that it can act as an elastodynamic cloak to incoming antiplane elastic waves. We deform the material so that its inner radius is significantly increased (to r_1) but its outer radius R_2 remains unchanged. The deformed cylindrical annulus can then slot into the existing cylindrical cavity region within the unbounded (unstressed) domain. We choose μ and ρ so that subsequent waves satisfy the necessary continuity conditions on $r = R_2$.

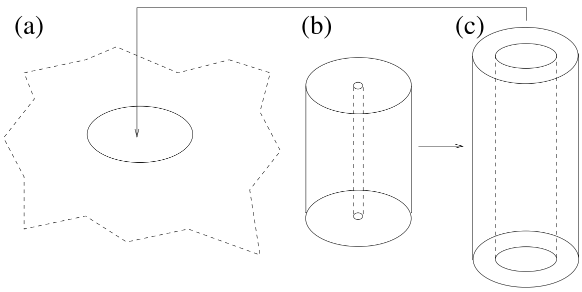

Figure 1: The incompressible neo-Hookean cylindrical annulus in (b) is pre-stressed as depicted in (c). This annulus then creates a cloak when slotted into a cylindrical cavity in an unbounded elastic medium (as illustrated in (a)).

The constitutive behaviour of an incompressible neo-Hookean material is described by the strain energy function [10]

$$\mathcal{W} = \frac{\mu}{2}(\lambda_r^2 + \lambda_\theta^2 + \lambda_z^2 - 3) \tag{5}$$

where λ_j, $j = r, \theta, z$ are the radial, azimuthal and axial principal stretches of the large deformation to ensue. We consider the initial deformation of the cylindrical annulus domain as depicted in Fig. 1. Since the material is incompressible, the deformation is induced *either* by applying a uniform axial stretch L *or* a radial pressure difference $p_o - p_i$ where p_o and p_i denote the pressures applied to the outer and inner face of the cylindrical annulus respectively. The ensuing deformation is described via the relations

$$R = R(r), \qquad \Theta = \theta, \qquad Z = z/L, \tag{6}$$

where (R, Θ, Z) and (r, θ, z) are cylindrical polar coordinates in the undeformed and deformed configurations. Note the convention introduced in (6), i.e. that upper case variables correspond to the undeformed configuration whilst lower case corresponds to the deformed configuration. This is analogous to the notation used for untransformed and transformed configurations in (2).

The principal stretches for this deformation are

$$\lambda_r = \frac{\mathrm{d}\,r}{\mathrm{d}\,R} = \frac{1}{R'(r)}, \quad \lambda_\theta = \frac{r}{R(r)}, \quad \lambda_z = L. \tag{7}$$

For an incompressible material $\lambda_r \lambda_\theta \lambda_z = 1$, implying

$$R(r) = \sqrt{L(r^2 + M)}, \tag{8}$$

where $M = R_2^2(L^{-1} - 1)$ is a constant determined by imposing that the outer wall of the cylindrical annulus remains fixed, i.e. $R(R_2) = R_2$. The deformation (8) is easily inverted to obtain $r(R)$. Given incompressibility and the fixed outer wall of the annulus, in order to induce this deformation we may either (i) prescribe the axial stretch L which then determines the deformed inner radius r_1 and the radial pressure difference required to maintain the deformation *or* (ii) prescribe the radial pressure difference which then determines the deformed inner radius r_1 and the axial stretch L.

We shall discuss the radial pressure difference shortly but either way we can obtain L and thus feed this into (8). Imposing the requirement that $R(r_1) = R_1$ and using the form of M gives rise to the useful relation

$$L = \frac{R_2^2 - R_1^2}{R_2^2 - r_1^2}. \tag{9}$$

The Cauchy stress for an incompressible material is [10]

$$\mathbf{T} = \mathbf{F}\frac{\partial \mathcal{W}}{\partial \mathbf{F}} + Q\mathbf{I}, \tag{10}$$

where $\mathcal{W}$ is the neo-Hookean strain energy function introduced in (5), $\mathbf{F}$ is the deformation gradient, $\mathbf{I}$ is the identity tensor and Q is the scalar Lagrange multiplier associated with the incompressibility constraint.

Only diagonal components of the Cauchy stress are non-zero, being given by (no sum on the indices)

$$T_{jj} = \mu_j(r) + Q \tag{11}$$

for $j = r, \theta, z$, where

$$\mu_r(r) = \frac{\mu^2}{L^2}\frac{1}{\mu_\theta(r)} = \frac{\mu}{L}\left(\frac{r^2 + M}{r^2}\right), \qquad \mu_z = L^2\mu. \tag{12}$$

The second and third of the static equations of equilibrium $\mathrm{Div}\,\mathbf{T} = \mathbf{0}$ (where Div signifies the divergence operator in the deformed configuration) merely yield $Q = Q(r)$. The remaining equation

$$\frac{\partial T_{rr}}{\partial r} + \frac{1}{r}(T_{rr} - T_{\theta\theta}) = 0, \tag{13}$$

can be integrated using (11)-(12) to obtain $Q(r)$. Writing $T_{rr}\big|_{r=R_2} = -p_o, T_{rr}\big|_{r=r_1} = -p_i$ we find

$$\frac{L(p_i - p_o)}{\mu} = \frac{1}{2L}\left(1 - \frac{R_1^2}{r_1^2}\right) + \log\left(\frac{r_1}{R_1}\right). \tag{14}$$

Given L and thus r_1 via (9), this equation prescribes the required pressure difference.

Now assume that the cylindrical annulus has been pre-stressed in an appropriate manner and slotted into the unbounded elastic material with perfect bonding at $r = R_2$. We consider wave propagation in this medium given a time-harmonic antiplane line source located at (R_0, Θ_0) with $R_0 > R_2$. In $r > R_2$ the antiplane wave with corresponding displacement which we shall denote by $w(r, \theta)$, is again governed by (1). In the region $r_1 \le r \le R_2$, the wave satisfies a different equation since this annulus region has been pre-stressed according to the deformation (6) and (8). We can obtain the governing equation using the theory of small-on-large [10]. It was shown in [5] that the wave in this region satisfies (3) but now where $\mu_r(r)$ and $\mu_\theta(r)$ are defined in (12) and $d(r) = \rho$, and note that we have made the necessary corrections in order to include the axial stretch L which was not considered in [5]. Note in particular that the density is homogeneous inside the cloak region.

In order to solve the scattering problem let us now introduce the identity mapping for $r > R_2$ and

$$R^2 = L(r^2 + M), \ \ \Theta = \theta, \ \text{for } r_1 \le r \le R_2 \tag{15}$$

which corresponds to the actual physical deformation (8). Finally define $W(R, \Theta) = w(r(R), \theta(\Theta))$. It is then straightforward to show that the equation governing wave propagation in the entire domain $R \ge R_1$ is (1), *provided that we choose* $\mu = L\mu_0$ *and* $\rho = L\rho_0$. These relations ensure that the wavenumbers in the exterior and cloak regions are the same and they also maintain continuity of traction on $R = R_2$. Furthermore since (15) corresponds to the actual deformation, the inner radius r_1 maps back to R_1. Therefore with the appropriate choice of cloak material properties, the scattering problem in the undeformed and deformed configurations are equivalent. We can therefore solve the equation in the undeformed configuration and then map back to the deformed configuration to find the physical solution. Decomposing the solution into incident and scattered parts $W = W_i + W_s$, we have $W_i = \frac{C}{4i\mu_0}\mathrm{H}_0(KS)$ where we have defined the wavenumber K via $K^2 = \rho_0\omega^2/\mu_0$ and $S = \sqrt{(X - X_0)^2 + (Y - Y_0)^2}$. Here $\mathrm{H}_n = \mathrm{H}_n^{(1)}$ is the Hankel function of the first kind of order n. The scattered field is written in the form [5]

$$W_s(R) = \sum_{n=-\infty}^{\infty} (-i)^n a_n \mathrm{H}_n(KR) e^{in(\Theta - \Theta_0)}. \tag{16}$$

Satisfaction of the traction free boundary condition on $R = R_1$ gives a_n. We want the wave field with respect to the *deformed* configuration, so we map back in order to find $w = w_i + w_s$. The incident wave is most conveniently determined by using Graf's addition theorem in order to distinguish between the regions $r < R_0$ and $r > R_0$, as was described in [5]. The incident and scattered fields are then, respectively,

$$w_i(r) = \frac{C}{4i\mu_0} \sum_{n=-\infty}^{\infty} e^{in(\theta - \theta_0)} Y_i(r) \tag{17}$$

where

$$Y_i(r) = \begin{cases} \mathrm{H}_n(KR_0) J_n(K\sqrt{L(r^2 + M)}), & r_1 \le r < R_2, \\ \mathrm{H}_n(KR_0) J_n(Kr), & R_2 \le r < R_0, \\ \mathrm{H}_n(Kr) J_n(KR_0), & r > R_0, \end{cases} \tag{18}$$

and

$$w_s(r) = -\frac{C}{4i\mu_0} \sum_{n=-\infty}^{\infty} e^{in(\theta - \theta_0)} \frac{\mathrm{J}_n'(KR_1)}{\mathrm{H}_n'(KR_1)} \mathrm{H}_n(KR_0) Y_s(r) \tag{19}$$

where

$$Y_s(r) = \begin{cases} \mathrm{H}_n\left(K\sqrt{L(r^2 + M)}\right), & r_1 \le r < R_2, \\ \mathrm{H}_n\left(Kr\right), & r \ge R_2. \end{cases} \tag{20}$$

The key to cloaking is to ensure that the scattered field is small compared with the incident field, i.e. $a_n \ll 1$. Note from (20) that a_n are solely dependent on the initial annulus inner radius R_1 (and source distance R_0) but are *independent* of the deformed inner radius r_1. Therefore we must choose R_1 such that $KR_1 \ll 1$ which will ensure negligible scattering. We illustrate with some examples in Fig. 2, showing that the "pre-stress" cloak appears to work well.

Let us now move onto a more general theory for compressible hyperelastic materials and the general elastodynamic case.

3 A general hyperelastic cloaking theory

It was fortuitous in the above theory that an incompressible neo-Hookean material gave us exactly the correct incremental behaviour under pre-stress in order to generate cloaks for antiplane waves. This prompts the question, for general elastodynamics in two and three dimensions, does there exist a hyperelastic material that can act similarly? In order to answer this question we must consider the formal equivalence of transformation elasticity and the equations of small on large.

Thus transformation elasticity take the Navier-Lamé equations

$$\frac{\partial}{\partial X_i}\left(C^0_{ijk\ell}\frac{\partial u^0_\ell}{\partial X_k}\right) + \rho_0\omega^2 u^0_j = 0 \tag{21}$$

for some homogeneous material properties $C^0_{ijk\ell}$ and ρ_0 and applies the mapping $\mathbf{x} = \chi_0(\mathbf{X})$ so that the governing equations become

$$\frac{\partial}{\partial x_i}\left(C^*_{ijk\ell}\frac{\partial u^*_\ell}{\partial x_k}\right) + \rho_*\omega^2 u^*_j = 0 \tag{22}$$

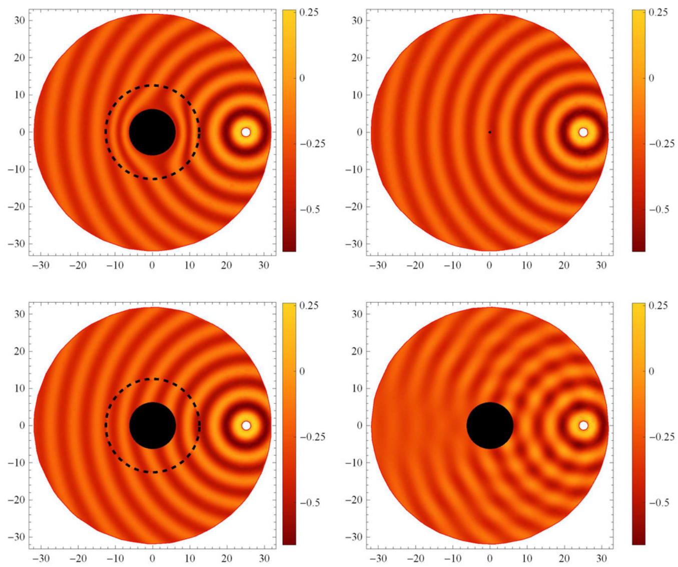

Figure 2: Cloaking of antiplane shear waves. Line source is located at $Kr = KR_0 = 8\pi$, $\Theta_0 = 0$, shown as a white circle. **Upper left**: A region of (nondimensionalized) radius $Ka = 2\pi$ is cloaked using a classic linear elastic cloak $g(R) = r_1 + R(\frac{R_2 - r_1}{R_2})$ in $2\pi \le Kr \le 4\pi$. **Upper right**: Scattering from a cavity of radius $KA = 2\pi/20$ in an unstressed medium. **Lower left**: A "pre-stress" cloak in $2\pi \le Kr \le 4\pi$ generated from an annulus with initial inner radius $KR_1 = 2\pi/20$. **Lower right**: Scattering from a cavity with radius $KA = 2\pi$ in an unstressed medium. Scattering and the shadow region presence in the latter is significant, as compared with that for an equivalent sized cavity for the "pre-stress" cloak.

where $\mathbf{u}^*(\mathbf{x}) = \mathbf{u}^0(\mathbf{X}(\mathbf{x}))$, $\rho_* = \rho_0/J_0$, $C^*_{ijk\ell} = F^0_{im}F^0_{kn}C^0_{mjn\ell}/J_0$ and $\mathbf{F}^0 = \mathrm{Grad}\chi_0$, $J_0 = \mathrm{Det}\mathbf{F}^0$. Thus for cloaking, as in the antiplane case considered above, a singular mapping χ_0 can be chosen such that the origin is mapped to a finite radius say r_1 whilst some radius further out, say R_2 remains fixed. Thus the cloak is the region $r \in [r_1, r_2]$ with $r_2 = R_2$. However we see that in general $C^*_{ijk\ell} \neq C^*_{ij\ell k}$. Thus the "cloak" is required to be an inhomogeneous Cosserat-type material.

Whereas the above refers to an "imaginary" transformation that allows the determination of the cloak properties, let us now consider an actual *physical* deformation (pre-stress) of a hyperelastic material with initial density $\bar{\rho}$ and constitutive behaviour governed by strain energy function (SEF) $\mathcal{W}$. As in [5] this is taken to be a pre-stress such that an initially small cavity with radius R_1 is inflated to a cavity with radius $r_1 > R_1$. The outer cloak radius is R_2. The deformation gradient is $\mathbf{F} = \mathrm{Grad}\chi$ where $\mathbf{x} = \chi(\mathbf{X})$ and the small-on-large equations governing wave propagation through this pre-stress material are

$$\frac{\partial}{\partial x_i}\left(M_{ijk\ell}\frac{\partial u_\ell}{\partial x_k}\right) + \rho\omega^2 u_j = 0 \qquad (23)$$

where $\rho = \bar{\rho}/J$ and $M_{ijk\ell} = (1/J)F_{im}F_{kn}\partial^2\mathcal{W}/\partial F_{jm}\partial F_{\ell n}$ with $J = \mathrm{Det}\mathbf{F}$. Ensuring the transformed and small-on-large equations are equivalent requires $\mathbf{u} = \mathbf{u}^*$, $\rho = \rho_*$ and $M_{ijk\ell} = C^*_{ijk\ell}$. It transpires that this gives a restriction on the SEF which for isotropic materials is required to be the so-called semi-linear SEF: $\mathcal{W} = (\bar{\lambda}/2)(\mathrm{tr}(\mathbf{U} - \mathbf{I}))^2 + \bar{\mu}\mathrm{tr}(\mathbf{U} - \mathbf{I})^2$ where $\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$ and $\bar{\lambda}, \bar{\mu}$ are the isotropic elastic moduli of the hyperelastic material. The deformation $r = r(R)$ can be determined explicitly but the invariance places the restriction that $r_1/R_1 < d/(d-1)$ where d is the dimension. Thus in 2D, for the examples that follow we are restricted by $r_1 < 2R_1$. We take the upper limit so that we have an "optimal" hyperelastic cloak of semilinear form.

Considering examples in 2D and defining the shear wavenumber via $K_s^2 = \omega^2\bar{\rho}/\bar{\mu}$, in Figure 3 we illustrate reduction in SH wave scattering from a cylindrical region by using a cloak generated by pre-stress, where a source is located at $K_sR_0 = 8\pi$, $\Theta_0 = 0$. Top left image shows the scattered field from a cavity of (scaled) radius $K_sr_1 = 2\pi$ whereas the top right is with the presence of the cloak. In particular note that the scattered field is far more isotropic than without the cloak. The plots underneath show the scattering cross-section (SCS) without (solid) and with (dashed) a cloak (left) and percentage reduction in SCS (right). Although the effectiveness of the cloak is reduced by the restriction $r_1 < 2R_1$, we see a significant reduction in scattering by employing the cloak. Similar effects are seen in the in-plane compressional-shear (P/SV) elastodynamic problem as illustrated in figure 4 where we have defined the compressional wavenumber $K_p^2 = \omega^2\bar{\rho}/(\bar{\lambda} + 2\bar{\mu})$.

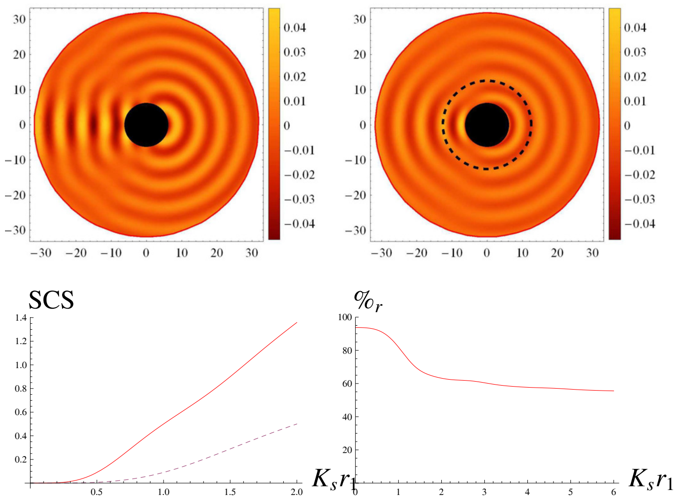

Figure 3: Top: Illustrating the scattered SH wave field of a cavity without (left) and with (right) a cloak. Bottom: Scattering cross-section (left) without (solid) and with (dashed) a cloak and percentage reduction in SCS (right) plotted against scaled cavity radius K_sr_1 where K_s is the shear wavenumber.

4 Conclusion

In conclusion, we have shown how a finite cloak can be generated for elastic waves by employing nonlinear pre-stress of a hyperelastic material.

In the first instance we illustrated how an incompressible neo-Hookean hyperelastic material could be used in order to cloak a region from antiplane waves. In this instance

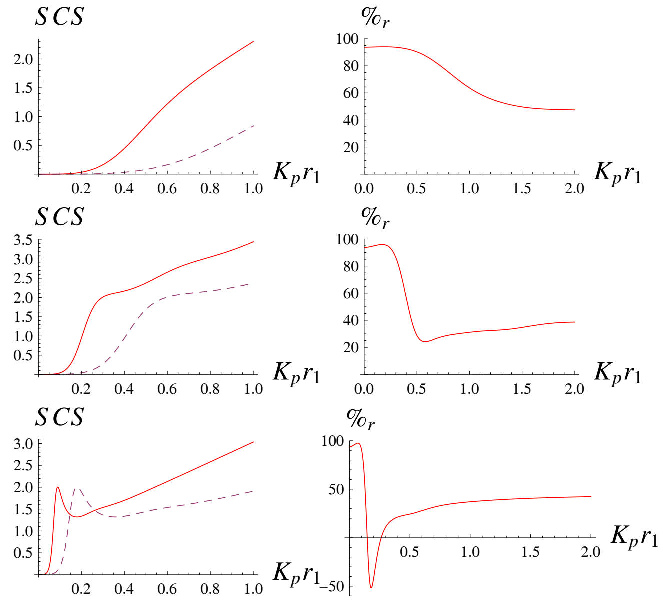

Figure 4: Scattering cross-section (left) without (solid) and with (dashed) a cloak and percentage reduction in SCS (right) plotted against scaled cavity radius $K_p r_1$ where K_p is the compressional wavenumber. From top to bottom, the Poisson ratio of the medium is $\nu = 1/3, 7/15$ and $49/99$. In general although perfect cloaking is not achieved, a significant reduction in scattering is achieved by using a hyperelastic cloak especially at low frequencies. Note that for $\nu = 49/99$ the local maximum in scattering cross section results in a narrow range of values of $K_p r_1$ where the cloak *increases* scattering.

the performance of the cloak is limited only by the size of the initial radius of the cylindrical cavity inside the annulus region. The anisotropic, inhomogeneous material moduli in the cloaking region, defined by (12), are induced naturally by the pre-stress and therefore exotic metamaterials are not required. Dispersive effects, which naturally arise in metamaterials due to their inherent inhomogeneity at some length scale, will not be present in the pre-stress context and we also note that the density of the cloak is homogeneous. In order to achieve the required pre-stress, a radial pressure difference is required across the cylindrical annulus. It would be inconvenient to prescribe p_o on the outer face. However, since we only need a pressure *difference* we can prescribe p_i with $p_o = 0$, ensuring the prescribed deformation and eliminating this difficulty. The incompressible neo-Hookean model is an approximation to reality, holding in general for rubber-like materials and moderate deformations. If the material is *not* neo-Hookean, invariance of the scattering coefficients is not guaranteed in general and therefore similar exact results will not hold. However it would be of interest to ascertain whether scattering from inflated cavities in other hyperelastic pre-stressed media is still significantly reduced as compared with an equivalent sized cavity in an unstressed medium.

We then pursued an equivalence between the full transformation elastodynamic equations and those of small-on-large. Specifically, the semilinear strain energy function uniquely yields the correct incremental moduli required for transformation of isotropic elasticity. The connection between the two theories is that the transformation equals the finite deformation. The fact that the pre-stress must be in a state of equilibrium places a constraint on the type of transformations allowed. Specifically, they are limited by the condition that $r_1 < 2R_1$ in 2D. This implies that the actual size of a cylindrical target can be increased in area by a factor of 4, its radius by factor of two, without any change to the scattering cross-section. The restricted form of the transformation is not surprising considering the fact that the theory can simultaneously control more than one wave type, in contrast to acoustics. It was shown that a significant reduction in the scattering cross-section from the cavity occurs, as compared with scattering from a cavity of the same radius in an undeformed medium. This effect is particularly striking at low frequencies and for small Poisson ratios.

The equivalence of transformation elasticity and small-on-large theory provides a unique and potentially realizable solution, although with a limited range of transformations allowed.

Acknowledgments

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