Attack transients in a clarinet model with time-varying blowing pressure

B. Bergeot[a], A. Almeida[a], C. Vergez[b] and B. Gazengel[a]

[a]Laboratoire d'acoustique de l'université du Maine, Bât. IAM - UFR Sciences Avenue Olivier Messiaen 72085 Le Mans Cedex 9
[b]LMA - CNRS (UPR 7051), 31 chemin Joseph-Aiguier, 13402 Marseille, Cedex 20, France
baptiste.bergeot.etu@univ-lemans.fr

Reed instruments are modeled as self-sustained oscillators driven by the pressure inside the mouth of the musician. A set of non-linear equations connects the control parameters (mouth pressure, lip force) to the system output, usually considered as the mouthpiece pressure. Clarinets can then be studied as dynamic systems, their steady behavior being dictated uniquely by the values of the control parameters. Considering the resonator as a lossless straight cylinder is a dramatic yet common simplification that allows for simulations using non-linear iterative maps. Many important aspects such as the kind of regime (static, oscillating), values of amplitude and periodicity have been predicted from such an approach. However, the existing studies focus mainly on the steady state, disregarding important features such as the attack transient. This presentation discusses transient behavior of these simplified clarinet models when the control parameters follow simple laws of variation with time. In this case, unexpected behavior can occur, such as bifurcation delays, meaning that oscillations do not start when the mouth pressure reaches the threshold value predicted by static bifurcation theory. This paper presents a first analytical/numerical study of bifurcation delay in clarinet-like instruments.

1 Introduction

One of the interests of mathematical models of musical instruments is to be able to predict certain characteristics of the produced sound given the gesture performed by the musician. In the case of a clarinet for instance, the amplitude, frequency or spectral content (the sound parameters) can be to a certain extent, determined as a function of the blowing pressure and lip force applied to the reed (the control parameters). A basic model, such as the one introduced by Mc. Intyre *& al.* [1] allows to compute the amplitude of the oscillating resonator pressure from the knowledge of these two control parameters, giving results that follow the major tendencies observed in experiments.

However, most studies of these models are restricted to a steady state analysis of the oscillation. They focus on the asymptotic amplitude regardless of the history of the system. In terms of the timbre perceived by a listener, this is an incomplete view, since the transients of the sound are probably as important as the characteristics of the steady state itself. Ideally, it would be interesting to predict the transient characteristics of the sound for a particular time evolution of the control parameters.

This work focuses precisely on the transient characteristics of a very simplistic model [2] when one of the parameters (the blowing pressure) increases with time. In these conditions, it is known from the so-called *bifurcation delay* theory that the beginning of the oscillation can be considerably delayed [3]. A similar effect is observed in simulations [4] and experiments [5] on the clarinet, although not to the same extent as predicted by the dynamic theory.

This article starts by summarising the basic model of the clarinet. The theoretical results of dynamic bifurcation theory are then applied to this model, and finally we put the theoretical results in perspective by comparing them to numerical simulations and analysing their sensitivity to numerical precision and the slope of the mouth pressure increase.

2 Elementary model of a clarinet

This model divides the instrument into two elements: the exciter and the resonator. The exciter is modeled by a nonlinear function F also called nonlinear characteristic of the exciter. The resonator (the bore of the instrument) is described by its reflection function $r(t)$.

In the case of a clarinet the coupling between the two elements allows to compute the state of the instrument throughout all values of time t. The state of the instrument model can be fully described by two variables: the pressure $p(t)$ inside the mouthpiece and the flow $u(t)$ created by the pressure imbalance between the mouth and the bore input.

The solutions $p(t)$ and $u(t)$ depend on the control parameters: γ representing the mouth pressure and ζ which characterizes the intensity of the flow. The nonlinear characteristic is provided by the Bernoulli equation describing the flow in the reed channel [6, 7].

Mathematical analysis of this model starts off from the extreme simplification of considering a straight, lossless (or losses independent of frequency) resonator and the reed as an ideal spring [2, 8, 9, 10, 11, 12]. With these assumptions, the reflection function becomes a simple delay with sign inversion. Using the variables p^+ and p^- (outgoing and incoming waves respectively) instead of the variables p and u, the system can be simply described by an iterated map [2] :

$$p_n^+ = G_\gamma\left(p_{n-1}^+\right). \tag{1}$$

The iteration function G_γ is obtained by transforming the nonlinear characteristic F. An explicit expression was determined by Taillard [13] and it is plotted in Figure 1 for a particular value of the parameters γ and ζ. This function (like the function F) depends on the control parameter γ representing the mouth pressure. The time step n corresponds to the round trip travel time $\tau = 2l/c$ of the wave with velocity c along the resonator of length l.

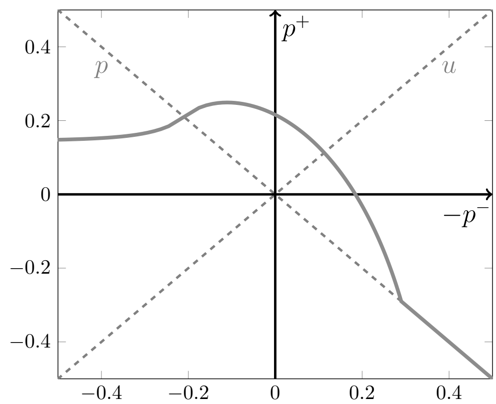

Figure 1: Iteration function G_γ for $\gamma = 0.42$ and $\zeta = 0.6$.

This simplistic model is certainly unable to describe or predict the exact harmonic content of the sound, or the influences of such important details as the reed geometry and composition or the vocal tract of the player. However, using the universal properties of the iterated maps [14], useful information about the instrument behavior can be drawn from the study of the iteration function. So far, these studies come from the *static* bifurcation theory, which assumes that the control parameter γ is constant. For example, it is possible to determine the steady state of the system as a function of the parameter γ, and to plot a bifurcation diagram. The oscillation threshold γ_t is, for a lossless model:

$$\gamma_t = \frac{1}{3}, \tag{2}$$

For all values of the control parameter γ below γ_t the series p_n^+ converges to a single value p^{+*} corresponding to the solution of $p_n^+ = p_{n-1}^+$, hence of $p^{+*} = G_\gamma(p^{+*})$. p^{+*} is the fixed point of G_γ. The fixed point depends on the value of γ according to:

$$p^{+*}(\gamma) = \frac{\zeta}{2}(1-\gamma)\sqrt{\gamma}. \tag{3}$$

When the control parameter γ exceeds γ_t the fixed point of G (the static regime) becomes unstable and the steady state becomes a 2-valued oscillating regime. Figure 2 shows an example of the bifurcation diagram with respect to the variable p^+.

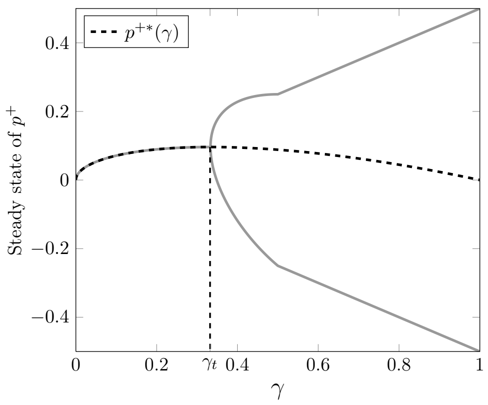

Figure 2: Graphical representation of the static bifurcation diagram for $\zeta = 0.5$.

To sum up, most of the studies using the iterated map approach, with the *static* bifurcation theory, are restricted to the steady state. It can predict the asymptotic (or *static*) behavior of an ideal clarinet as a function of the mouth pressure when it is constant. In other words, static results are obtained by choosing a value of γ, letting the system relax to its final state, and repeating the procedure for each value of γ. This procedure avoids the phenomenon of *bifurcation delay* which is observed in numerical simulations, for instance (section 4).

3 Slowly time-varying mouth pressure

In this section we investigate the consequences of a linear growth of the control parameter γ (the parameter ζ is always constant) on the behavior of the ideal clarinet, henceforth described by the following system of equations:

$$\begin{cases} p_n^+ = G\left(p_{n-1}^+, \gamma_n\right) & \text{(4a)} \\ \gamma_n = \gamma_{n-1} + \epsilon. & \text{(4b)} \end{cases}$$

A slowly varying parameter implies that $\epsilon \ll 1$. We highlight the phenomenon of bifurcation delay, Section 3.1, and explain how it can be analytically predicted, Section 3.2.

3.1 Bifurcation delay

In the static case (γ is a constant) and in the dynamic case (γ increases linearly), when $\gamma > \gamma_t$ the system is unstable. Therefore, the first intuitive hypothesis about the dynamic behavior of the system might be to say that the orbit of the series p_n^+ (described by the system (4)) follows the static bifurcation diagram. Simulations of the system (4) undermine this hypothesis. Indeed, Figure 3 shows that when γ varies the bifurcation point is shifted from γ_t to a value γ_{dt} called *dynamic* oscillation threshold. The difference between γ_t (called now *static* oscillation threshold) and γ_{dt} is the bifurcation delay.

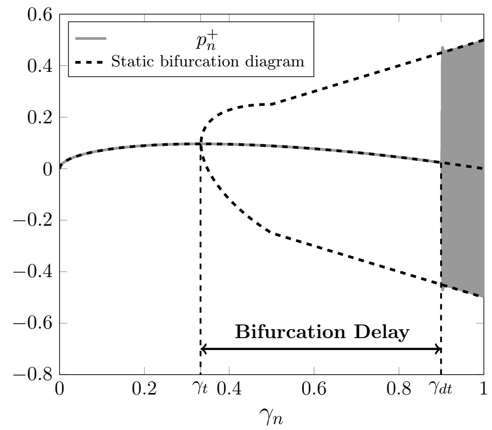

Figure 3: Representation of the series p_n^+ for $\epsilon = 10^{-4}$ and $\zeta = 0.5$, comparison between the series p_n^+ and the *static* bifurcation diagram as functions of γ_n.

The phenomenon of bifurcation delay can be interpreted as an "accumulation" of stability during the time for which $\gamma_n < \gamma_t$, when the system is stable [15, 16]. The accumulation of stability has to compensated by going beyond the static bifurcation point γ_t for a certain time.

Static bifurcation theory is not sufficient to predict the bifurcation delay. *Dynamic* bifurcation theory [3] is able to produce some results about the behavior of such systems.

3.2 Analytical results from dynamic bifurcation theory

The behavior of the dynamical system (4) can be studied mathematically thanks to the *dynamic* bifurcation theory based on the non-standard analysis [3, 17, 18]. Non-standard analysis is a branch of mathematics that allows to deal rigorously with the infinitesimal numbers, in our case the infinitesimal number considered is the slope ϵ, the increase rate of a parameter (for instance γ) per iteration. Dynamic bifurcation theory allows to describe the asymptotic behavior of the system when $\epsilon << 1$ in analytical cases. Furthermore, these results imply that all the calculations are performed exactly (without any added noise or with infinite precision). We shall return later to this point (Section 4).

Even if the non-standard analysis is very technical, the results it produces may have simple expressions. One of them is the prediction of the dynamic oscillation threshold γ_{dt}. Reminding that the nonlinear function G is defined by the relation $p^+ = G(-p^-, \gamma)$, the dynamic oscillation threshold $\gamma_{dt}$is a solution of the following equation:

$$\int_0^{\gamma_{dt}} \ln \left| -\frac{\partial G}{\partial p^-} \left[p^{+*}(u), u \right] \right| du = 0. \quad (5)$$

The demonstration of equation (5) is made in [3]. The numerical solution of this equation is plotted in Fig. 4 as a function of the parameter ζ.

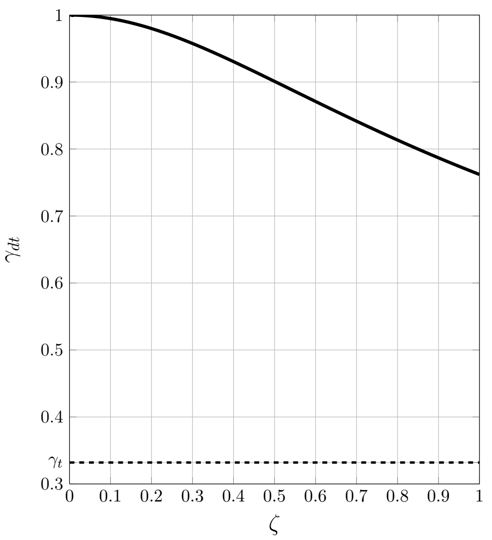

Figure 4: Graphical representation of the dynamic oscillation threshold γ_{dt} (solid line) and of static the oscillation threshold γ_t (dashed line) as functions of the parameter ζ.

Unlike the static oscillation threshold γ_t, the dynamic one depends on ζ. More precisely, we can see that the dynamic oscillation threshold tends to 1 when ζ tends to 0 and then decreases when ζ increases, but it is always far above $\gamma_t = 1/3$.

The second (and counter-intuitive) result about the dynamic oscillation threshold is that it is independent of the slope ϵ. However, as said before, equation (5) is valid if $\epsilon << 1$ and if there is no noise in the system. No noise in the system means that if we want to compare analytical results with simulations, we have to use an "infinitely" high precision[1]. The orbit of p_n^+ plotted in Figure 3 is obtained with *mpmath*, the arbitrary precision library of *Python*, using a precision equal to 5000. Above this precision, γ_{dt} does not vary considerably. Comparing Figure 3 and Figure 4, we can see that γ_{dt} seams to be predicted by equation (5), i. e. $\gamma_{dt} = 0.9$.

In section 4 we perform a more quantitative comparison between the analytical prediction of the dynamic oscillation threshold γ_{dt} and its estimation made on simulation.

4 Comparison between analytical results and simulations

The purpose of this section is to evaluate the precision and of values of ϵ required for a good agreement between analytical results and simulations. We study the influence of the precision used and of the value of the slope ϵ on the bifurcation delay.

[1]The precision is the number of decimal digits used by the computer.

4.1 Influence of the precision

This paragraph shows that the bifurcation delay has an exponential sensitivity. It thus depends on the precision used in the simulations.

To highlight this dependence, we estimate the dynamic oscillation threshold on the simulations of the system (4). The dynamic oscillation threshold estimation is defined as the value of γ (noted γ_{osc}) when the series p_n^+ begins to oscillate. Results are plotted for different precisions and compared to the analytical value in Figure 6.

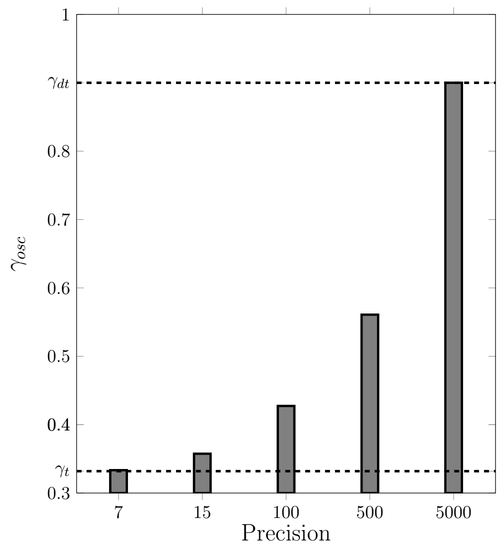

Figure 5: Graphical representation of γ_{osc} for different precisions (prec. = 7, 15, 100, 500 and 5000) and for $\epsilon = 10^{-4}$. Results are also compared to analytical *static* and *dynamic* thresholds: γ_t and γ_{dt}.

The first observation we can do on Figure 5 is the very high dependence of γ_{osc} on the precision. We can also notice that all the values of γ_{osc} are between γ_t and γ_{dt}. For the lowest precision (prec. = 7) the bifurcation delay disappears and $\gamma_{osc} = \gamma_t$. The precision must be very high (prec. = 5000) to reach $\gamma_{osc} = \gamma_{dt}$. Therefore, γ_{dt} can be interpreted as the limit of the bifurcation delay when precision tends to infinity. In the cases with intermediate precisions (prec. = 15, 100 and 500) the bifurcation delay increases with the precision.

4.2 Influence of the mouth pressure slope

To study the influence of the slope, γ_{osc} is plotted against ϵ in Figure 6, for different precision values.

As above, for the lowest precision (prec.=7) the bifurcation delay disappears when ϵ is sufficiently small. Indeed, γ_{osc} is constant and equal to γ_t. Then γ_{osc} recurs and increases with ϵ. The case with the highest precision (prec. =5000) simulates an analytic case which would correspond to an infinite precision. We can see that if ϵ is sufficiently small γ_{osc} is constant and equal to 0.9 as predicted by equation (5). Then γ_{osc} decreases when ϵ increases. In the cases with intermediate precisions (prec. = 15, 100 and 500) the curve of γ_{osc} is first increasing and then reached the curve corresponding to the higher precision.

The value of γ_{osc} also depends on the parameter ζ. The general form of graphic represented in Figure 6 is the same whatever the value of ζ. The value of γ_{osc} is just smaller

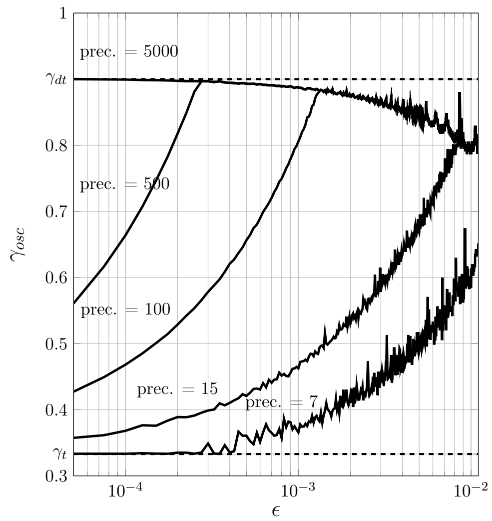

Figure 6: Graphical representation of γ_{osc} as a function of ϵ for $\zeta = 0.5$ and using five values for the precision. A logarithmic scale is used in abscissa.

when ζ increases, except in the lower left side of the graphic because γ_{osc} cannot be smaller than γ_t.

5 Conclusion

The dynamical study of a nonlinear system is a complement to the usual static study. It may predict phenomenons (bifurcation delay and dynamic bifurcation) not explained by the static study but observed by simulations and experiments. The phenomenon of bifurcation delay extends upon a vast operating rage of the model: $1/3 < \gamma_{dt} < 1$, depending of two parameters: the precision used in simulations and the slope of the linear growth of γ. Indeed, we saw, in dynamical case, that the system is very sensitive to this two parameters. This work allows to understand the difficulties encountered in studies aimed at comparing static oscillation threshold to experimental/numerical ones. A companion paper [5] presents a first experimental study of bifurcation delay in clarinet-like system. These preliminary experimental results confirm the tendencies observed in the present paper.

Acknowledgments

This work was done within the framework of the project SDNS-AIMV "Systèmes Dynamiques Non-Stationnaires - Application aux Instruments à Vent" financed by *Agence Nationale de la Recherche*.

References

[1] M. E. Mcintyre, R. T. Schumacher, and J. Woodhouse. On the oscillations of musical instruments. *J. Acoust. Soc. Am.*, 74(5):1325–1345, November 1983.

[2] C. Maganza, R. Caussé, and F. Laloë. Bifurcations, period doublings and chaos in clarinetlike systems. *EPL (Europhysics Letters)*, 1(6):295, 1986.

[3] A. Fruchard and R. Schäfke. Sur le retard à la bifurcation. In *International conference in honor of claude lobry*, 2007.

[4] Mérouane Atig, Jean-Pierre Dalmont, and Joël Gilbert. Saturation mechanism in clarinet-like instruments, the effect of the localised non-linear losses. *Appl. Acoust.*, 65(12):1133–1154, 2004.

[5] B. Bergeot, C. Vergez, A. Almeida, and B. Gazengel. Measurment of attack transients in a clarinat driven by a ramp-like varying pressure. In *11ème Congrès Français d'Acoustique and "2012 Annual IOA Meeting*, Nantes, France, April 23th-27th 2012.

[6] A. Hirschberg, R. W. A. Van de Laar, J. P. Maurires, A. P. J Wijnands, H. J. Dane, S. G. Kruijswijk, and A. J. M. Houtsma. A quasi-stationary model of air flow in the reed channel of single-reed woodwind instruments. *Acustica*, 70:146–154, 1990.

[7] A. Hirschberg. *Mechanics of musical instruments*. chap. 7, p. 291-369. N° 355 in CISM courses and lectures, Springer-Verlag, New York, 1995.

[8] J. Kergomard. Elementary considerations on reed-instrument oscillations. *Mechanics of musical instruments*, by A. Hirschberg/ J. Kergomarg/ G. Weinreich. Volume 335 of *CISM Courses and lectures*:229–290., Springer-Verlag, Wien, 1995.

[9] J. Kergomard, J-. P. Dalmont, J. Gilbert, and P. Guillemain. Period doubling on cylindrical reed instruments. In *Proceeding of the Joint congress CFA/DAGA 04*, pages 113–114. Société Française d'Acoustique - Deutsche Gesellschaft für Akustik, 22th 24th March 2004, Strasbourg, France.

[10] S. Ollivier, J. D. Dalmont, and J. Kergomard. Idealized models of reed woodwinds. part 2 : On the stability of two-step oscillations. *Acta. Acust. Acust.*, 91:166–179, 2005.

[11] Jean-Pierre Dalmont, Joel Gilbert, Jean Kergomard, and Sebastien Ollivier. An analytical prediction of the oscillation and extinction thresholds of a clarinet. *J. Acoust. Soc. Am.*, 118(5):3294–3305, 2005.

[12] A. Chaigne and J. Kergomard. *Acoustique des instruments de musique*. Belin, 2008.

[13] P.-A. Taillard, J. Kergomard, and F. Laloë. Iterated maps for clarinet-like systems. *Nonlinear Dynamics*, 62:253–271, 2010. 10.1007/s11071-010-9715-5.

[14] M. J. Feigenbaum. The universal metric properties of nonlinear transformations. *J. Stat. Phy.*, 21(6):669–706, 1979.

[15] R. Kapral and P. Mandel. Bifurcation structure of the nonautonomous quadratic map. *Phys. Review A*, 32(2):1076–1081, 1985.

[16] J. R. Tredicce, G.L. Lippi, P. Mandel, B. Charasse, A. Chevalier, and B. Picqué. Critical slowing down at a bifurcation. *American Journal of Physics*, 72(6):799–809, 2004.

[17] A. Fruchard. Sur l'équation aux différences affine du premier ordre unidimensionnelle. *Ann. Inst. Fourier*, 46:139–181, 1996.

[18] A. Fruchard and R. Schäfke. Bifurcation delay and difference equations. *Nonlinearity*, 16:2199–2220, 2003.