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Time-harmonic acoustic scattering in a complex flow

J.F. Mercier[a], A.-S. Bonnet-Bendhia[a], F. Millot[b], S. Pernet[b] and E. Peynaud[b]

[a]POEMS-CNRS/ENSTA/INRIA, 32 Boulevard Victor, 75015 Paris, France
[b]CERFACS, 42 Avenue Gaspard Coriolis 31057 Toulouse Cedex 1
jean-francois.mercier@ensta-paristech.fr

We are interested in the numerical simulation of time harmonic acoustic scattering in presence of a complex flow on an unstructured mesh. Galbrun's equation, whose unknown is the perturbation of displacement, is attractive, compared to the linearized Euler's equations, because it is close to a wave equation which allows the use of classical Lagrange Finite Element, and it is well adapted to take into account boundary conditions, like impedance or interface with an elastic structure. However, a direct discretization of Galbrun's equation with Lagrange Finite Elements leads to numerical troubles. We propose a method that allows both to obtain a stable numerical scheme and non-reflecting artificial boundary conditions. This method requires to introduce a new quantity related to hydrodynamic vortices which satisfies a convection equation. A hybrid numerical method is proposed, coupling Finite Element for Galbrun's equation and a Discontinuous Galerkin scheme for the convection equation. Several 2D numerical results are presented to show the efficiency of the method.

1 Introduction

The reduction of noise in aeronautics motivates an intensive research in aeroacoustics. In particular the radiation of the sound produced by aircraft engines is strongly influenced by the presence of the flow around the airplane. A more efficient numerical simulation of acoustic propagation would be a useful tool to improve noise reducing in planes or cars industry. The main difficulty when acoustic waves propagate in presence of a mean flow is the coupling between acoustics waves and hydrodynamic vortices.

Our objective is to develop a numerical method to solve the acoustic radiation in time-harmonic regime ($e^{-i\omega t}$), in an unbounded domain and in a quite general case in the sense that the geometry, and therefore the mean flow, can be complex. As a consequence, discretization methods written on an unstructured mesh will be privileged. The unknown is a small perturbation of a given flow, which naturally leads to consider linearized equations. Contrary to the classical case of acoustics in a fluid at rest, the obtained problem is vectorial because of the coupling between acoustics and hydrodynamics.

In the time domain most of the works mentioned in literature deal with Euler's linearized equations, using Finite Difference methods [1, 2, 3, 4, 5, 6] or discontinuous Galerkin methods [7, 8]. In the time-harmonic regime, an extra difficulty has to be faced: the treatment of unbounded domains is hard to take satisfactorily into account and requires to introduce some well-adapted boundary conditions around the calculation domain.

Up to our knowledge, only the potential case (when the flow and the source are irrotational) which leads to a Helmholtz like scalar equation has been completely handled [9]. In this case, acoustic perturbations are modelized by a scalar generalized Helmholtz equation : in particular, if the flow is uniform far from the source, the problem can be solved in a classical way, by coupling Finite Elements with modal [9] or with integral [10] representation of the far-field.

For an arbitrary flow, the problem is much more difficult to solve, due to the presence of hydrodynamic phenomena. In the sequel we explain why the Linearized Euler Equations do not seem to us well adapted to the time-harmonic regime. Then we show that the Galbrun approach, less usual than the Euler one, seems more adapted, in particular to deal with unbounded domains. We present our method, based on the introduction of Perfectly Matched Layers (PMLs) combined with a Finite Element discretization of an augmented formulation of Galbrun's equation. We show that the acoustic waves can be discretized using continuous Finite Elements whereas the vortices require the use of discontinuous elements. Numerical validations are then presented and the strengths of the Galbrun equation are highlighted: we illustrate why this model seems also more adapted than Euler's equations to take into account general boundary conditions, like lined walls or elastic boundaries.

2 Geometry and flow

We consider a general geometry including the presence of rigid obstacles and/or rigid walls confining the fluid. This leads to a complex flow circulating between the rigid boundaries (Figure 1).

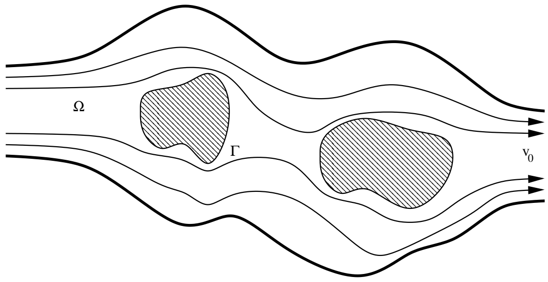

Figure 1: The complex geometry

We note Ω_∞ the domain filled with a compressible inviscid fluid and Γ_∞ the union of all the rigid boundaries. The flow is supposed stationary and homentropic (entropy is constant and uniform). It is characterized by its non uniform fields of velocity $\mathbf{v}_0$, density ρ_0, pressure p_0 and solves in Ω_∞ the stationary Euler Equations combined with the state law:

$$\begin{cases} \operatorname{div}(\rho_0 \mathbf{v}_0) = 0, \\ \rho_0 (\mathbf{v}_0 \cdot \nabla) \mathbf{v}_0 + \nabla p_0 = 0, \\ p_0 = \kappa \rho_0^\gamma. \end{cases} \tag{1}$$

where κ is a constant, $\gamma = c_p/c_v$ with c_v the specific heat capacity at constant volume and c_p the specific heat capacity at constant pressure. On the rigid boundaries:

$$\mathbf{v}_0.\mathbf{n} = 0 \quad (\Gamma_\infty), \tag{2}$$

where $\mathbf{n}$ denotes the exterior normal vector to Γ_∞.

A first difficulty, rarely mentioned in the literature, raises: although the system (1) is valid for a general compressible non-potential flow, its direct numerical resolution leads in general to an incompressible and potential flow which is often not physical. For instance it is difficult to capture recirculation areas behind obstacles. This is why in practice to deal with a non-potential flow, "academic" flows like parallel shear flows of the form $\mathbf{v}_0(x, y) = u_0(x)\mathbf{e}_y$ (shear layer or jet flow in particular) are considered. In more realistic geometries like the one in Figure 1, it appears to be necessary to add some viscosity to get a non-potential flow. Note that introducing viscosity implies that the flow does not satisfy (1) anymore. The relevance of neglecting the viscosity in the equations for the perturbations is up to our knowledge an open question and should be clarified.

Now we want to determine how acoustic perturbations propagate in this complex flow. The most popular model is the Euler equations. It has been proved to be efficient in the time domain and on structured meshes. We will show that it is not the case in the time harmonic regime and on unstructured meshes.

3 Resolution of Euler's equations: the difficulties

The flow is perturbed and the small perturbations of velocity $\mathbf{v}$ and of pressure p satisfy the Linearized Euler Equations:

$$\begin{aligned} \frac{D\mathbf{v}}{Dt} + (\mathbf{v}\cdot\nabla)\mathbf{v}_0 + \nabla\left(\frac{p}{\rho_0}\right) &= \mathbf{f}, \\ \rho_0\frac{D}{Dt}\left(\frac{p}{\rho_0 c_0^2}\right) + \operatorname{div}(\rho_0\mathbf{v}) &= 0, \end{aligned}$$

where $D/Dt = -i\omega + \mathbf{v}_0\cdot\nabla$, $\mathbf{f}$ is a source term compactly supported in Ω_∞ and where we have linearized the state law to get $p = c_0^2\rho$ with ρ the density and $c_0^2 = \gamma p_0/\rho_0$ the non uniform sound velocity.

Therefore the Linearized Euler Equations read naturally as two coupled transport equations. It is known that first order systems of equations are not adapted to a continuous Finite Element resolution and more sophisticated elements must be chosen. To illustrate this fact, let us focus on a model transport equation of the form:

$$\frac{D\psi}{Dt} = g, \quad \text{with} \quad \psi = 0 \quad \text{upstream the source.} \qquad (3)$$

Note that Euler's equations reduce exactly to this equation in the case of a uniform flow and a divergence-free source: then we can prove that $\operatorname{div}\mathbf{v} = 0$ thus $\mathbf{v} = \mathbf{curl}\,\psi$ and we get the transport equation if we note $\mathbf{f} = \mathbf{curl}\,g$. A direct discretization with Lagrange (continuous) Finite Elements of such transport problem does not lead to good results . Such discretization is performed in Figure 2 for a shear flow and spurious oscillations are clearly observed.

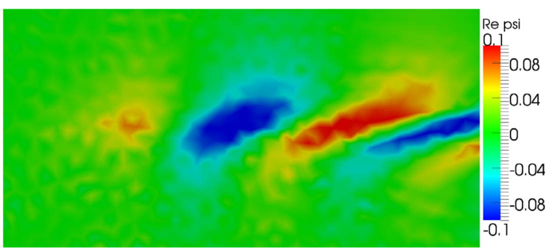

Figure 2: Numerical solution of Eq. (3) obtained by a discretization with Lagrange (continuous) Finite Elements

A discretization using a classical discontinuous Galerkin method is better adapted to take into account transport phenomena [11]. Such method based on Discontinuous Elements has been used to solve Linearized Euler's Equations with the choice of plane wave solutions on each element [12]. However it is numerically more expensive than continuous Finite Elements and in 3D it becomes numerically too costly. We have developped an alternative method, based on the Galbrun equation, whose main advantage is to allow the use of Lagrange Finite Elements.

4 A remedy: resolution of Galbrun's equation

4.1 The Galbrun equation

Galbrun's system [13] corresponds to a linearized model whose unknown $\mathbf{u}$ is the perturbation of the Lagrangian displacement, linked to the usual Euler unknowns by:

$$p = c_0^2\rho = -c_0^2\operatorname{div}(\rho_0\mathbf{u}) \quad \text{and} \quad \mathbf{v} = \frac{D\mathbf{u}}{Dt} - (\mathbf{u}\cdot\nabla)\mathbf{v}_0.$$

Plugging these expressions in linearized Euler's equations leads to the Galbrun equation:

$$\rho_0\frac{D^2\mathbf{u}}{Dt^2} - \nabla(c_0^2\operatorname{div}(\rho_0\mathbf{u})) + (\operatorname{div}\mathbf{u} + \mathbf{u}\cdot\nabla)\nabla p_0 = \mathbf{f} \quad (\Omega_\infty). \quad (4)$$

The normal component of the displacement vanishes on the rigid boundaries:

$$\mathbf{u}.\mathbf{n} = 0 \quad (\Gamma_\infty). \qquad (5)$$

Galbrun's model is more attractive than Euler's one: contrary to the Linearized Euler Equations, it does not involve any derivatives of the mean flow quantities. In particular it is more adapted to study a discontinuous shear layer. Moreover Galbrun's equation is a second order equation in time and in space, at first sight similar to more classical wave models and thus well adapted to a Lagrange Finite Element discretization. However although Galbrun's equation has been known for a long time, its numerical solution has always been problematic. Indeed it is not exactly a wave equation, since we do not have $\Delta\mathbf{u}$ but only $\nabla(\operatorname{div}\mathbf{u})$. A consequence is that the direct use of a Finite Element method to solve it leads to very bad results. Moreover the introduction of PMLs arround the calculation domain to deal with unbounded domain causes numerical troubles. Introducing the pressure as a new unknown, a mixed Finite Element scheme has been developed [14, 15], which has been checked to be stable in several applications. However this mixed approach does not help to deal satisfactorily with the convection of vortices in unbounded domain.

The method we propose to get rid of these difficulties is the so-called *augmentation* of Galbrun equation, based on the use of the identity $\nabla(\operatorname{div}\mathbf{u}) - \mathbf{curl}(\operatorname{curl}\mathbf{u}) = \Delta\mathbf{u}$.

4.2 The augmented Galbrun equation

The augmentation process consists in adding to the equation a term which does not change the value of the solution (the additional term vanishes for the solution) but which improves the mathematical properties of the equation. As a consequence, it is well-suited for a discretization by Lagrange Finite Element.

We consider the following "augmented" formulation:

$$\begin{aligned} &\rho_0\frac{D^2\mathbf{u}}{Dt^2} - \nabla(c_0^2\operatorname{div}(\rho_0\mathbf{u})) + \mathbf{curl}[\rho_0 c_0^2(\operatorname{curl}\mathbf{u} - \psi)] \\ &+(\operatorname{div}\mathbf{u} + \mathbf{u}\cdot\nabla)\nabla p_0 = \mathbf{f} \quad (\Omega_\infty), \end{aligned} \qquad (6)$$

where we have introduced a new unknown :

$$\psi = \operatorname{curl} \mathbf{u},$$

called here the "vorticity" (in the literature, the vorticity is usually defined as the curl of the Eulerian velocity). To get equivalence with the Galbrun equation we need to add the boundary condition:

$$\operatorname{curl} \mathbf{u} - \psi = 0 \quad (\Gamma_\infty). \tag{7}$$

It has been proved in [16] that ψ satisfies the following equation:

$$\frac{D^2\psi}{Dt^2} = -2\frac{D}{Dt}(\mathcal{B}\mathbf{u}) - C\mathbf{u} + \frac{1}{\rho_0}\operatorname{curl}\mathbf{f} \tag{8}$$

with

$$\mathcal{B}\mathbf{u} = \sum_{j=1}^{2} \nabla v_{0,j} \wedge \frac{\partial \mathbf{u}}{\partial x_j}.$$

The expression of $C\mathbf{u}$ is more complicated and is not mentionned here because in practice this term is negligible (in particular $C\mathbf{u}$ vanishes everywhere for a parallel shear flow). Let us point out that the equivalence requires the regularity of Γ_∞ (see for instance the remark 3.5 in [17]), and the treatment of reentrant corners still raises open questions of modelization. Moreover in the case of a mean flow without recirculation (closed streamlines) [18, 19], Eq. (6) and Eq. (8) constitute a well-posed problem.

Note that the ψ unknown can be eliminated for a slow flow [16]: we can then replace the exact non-local expression of ψ by a simple local formula. Indeed since D/Dt becomes simply $-i\omega$, Eq. (8) becomes the explicit relation:

$$\psi = \frac{2}{i\omega}(\mathcal{B}\mathbf{u}).$$

This low Mach approach has been validated in the case of both a potential and a parallel flow, for which reference solutions are available [16, 20].

Now we want to solve Eq. (6) and Eq. (8) using a Finite Element method.

4.3 Numerical solution of the coupled problem

4.3.1 The numerical scheme

As already mentioned in the first part, the convection equation (8) can not be solved with Lagrange Finite Elements. We have developped a hybrid numerical method for the solution, coupling Finite Element for Galbrun equation (6) and a Discontinuous Galerkin scheme for the convection equation (8). The same mesh is used to discretize $\mathbf{u}$ and ψ. For the Discontinuous Galerkin scheme, a penalized formulation and upwind fluxes have been chosen. Finally our formulation allows to easily increase the order of the Finite Elements.

4.3.2 The Perfectly Matched Layers

To compute the "outgoing" solution of the coupled problem (5, 6, 7, 8), we have chosen to introduce PMLs. This is a commonly used method in the literature, although it does not extend easily to acoustic propagation in presence of a flow, the difficulty being to handle simultaneously acoustic and hydrodynamic phenomena. Around the domain of interest $[-R, R] \times [-R, R]$ we introduce absorbing layers of width L (see Figure 3 for a rectangular calculation domain). The model in the PMLs involves a complex parameter α such that $\Re e(\alpha) > 0$ and $\Im m(\alpha) < 0$. It consists in modifying the operators according to the substitution:

$$\frac{\partial}{\partial x_i} \to \alpha_i(x)\frac{\partial}{\partial x_i},$$

with α_i defined by $\alpha_i(x) = 1$ if $|x_i| < R$ and $\alpha_i(x) = \alpha$ if $|x_i| > R$. For example, $\operatorname{div} \mathbf{u}$ becomes

$$\operatorname{div}_\alpha \mathbf{u} = \alpha_1(x)\frac{\partial u_1}{\partial x_1} + \alpha_2(x)\frac{\partial u_2}{\partial x_2}$$

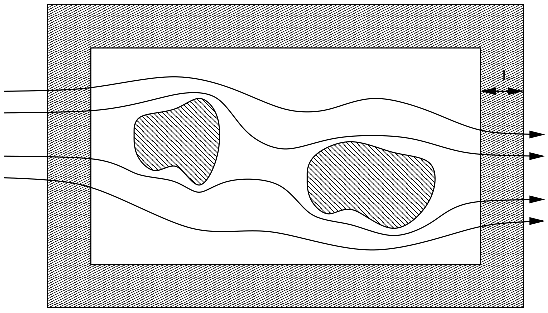

Figure 3: The PMLs

To select the "good" hydrodynamic solution (the causal one), we impose $\psi = 0 = D\psi/Dt$ on the vertical PML boundary at $x_1 = -R - L$ where the flow enters: this means that no vortices are convected by the flow from outside to inside the calculation domain. Note that no condition is required on the symetric side $x_1 = R + L$ where the flow exits the calculation domain. To impose such non symetric boundary conditions is natural in the Discontinuous Galerkin framework, it would not be the case if continuous Finite elements are used in a variational framework.

Remark: this is another interest of our method based on an augmented Galbrun equation: adding to the fact that the stability of a classical continuous Finite Element scheme is ensured, our method is well-adapted to the introduction of PMLs since the acoustic and hydrodynamic phenomena are solved separately. In particular the use of PMLs for the non-augmented Galbrun formulation leads to bad results because these layers are able to select the outgoing acoustic waves but are not able to deal satisfactory with the vortices. Indeed the PMLs are designed for wave phenomena, propagating in all directions, but not for transport phenomena, which propagate only in the direction of the flow.

5 Numerical illustrations

5.1 Academic tests

In this part, we present an academic numerical result obtained from the augmented Galbrun equation. We use the jet mean flow described on the left of Figure 4. On the right of Figure 4 is represented $\Re e(u_1)$ induced by a source whose support is delimited by a red circle. Two different kinds of structures corresponding to acoustic and hydrodynamic phenomena are clearly observed. The inclined lines correspond to the vortices, mainly produced by the source and convected by the flow. The acoustic waves correspond to the large patterns, associated to short wavelengths upstream and to large ones downstream (Doppler effect).

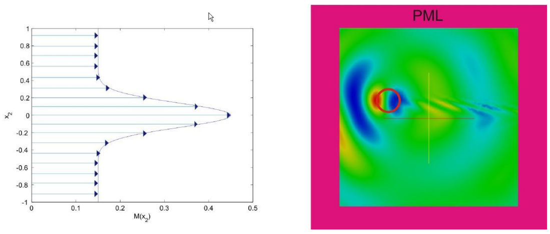

Figure 4: Numerical solution of of augmented Galbrun's equation for a jet flow

5.2 Domain of validity of the potential model

When the flow is potential $\mathbf{v}_0 = \nabla\varphi_0$, the perturbations are potential $\mathbf{v} = \nabla\varphi$ and the potential satisfies the wave equation:

$$\begin{aligned} \frac{D}{Dt}\left(\frac{1}{c_0^2}\frac{D\varphi}{Dt}\right) - \frac{1}{\rho_0}\,\mathrm{div}\,(\rho_0\nabla\varphi) &= \mathbf{f} \quad (\Omega_\infty), \\ \frac{\partial\varphi}{\partial n} &= 0 \quad (\Gamma_\infty). \end{aligned}$$

Of course these equations can be solved for any flow, in particular for a non-potential flow. Then this potential model is wrong but when $\mathrm{curl}(\mathbf{v}_0)$ is weak one could expect the results to be rather good. Since we have with the Galbrun equation an exact alternative model, we can check the validity of using a potential model for a non-potential flow. Figure 5 shows a comparison of the acoustic velocity $\Re e(\mathbf{v}_1)$ computed from the potential and the Galbrun models in presence of a non-potential flow (the jet flow described on Figure 4). The radiation of a quadripole source located above the shear layer is considered and the refraction effect of the jet core is highlighted. We clearly see the influence of the hydrodynamic phenomena on the wave directivity, effect that can not be caught by the potential model.

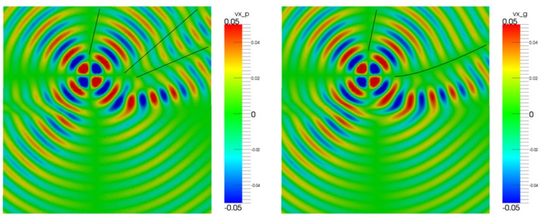

Figure 5: Numerical acoustic velocity: potential model (left) and Galbrun's model (right)

5.3 Treatment of non-rigid boundary conditions

The Galbrun approach allows a very simple treatment of the boundary conditions, which are generally expressed with respect to the displacement (rather than the pressure of the velocity). This applies in particular for lined boundaries or fluid-structure coupling.

5.3.1 Lined walls

In presence of lined walls the boundary condition due to Myers [21] reads:

$$p = -i\omega\beta\,\mathbf{u}\cdot\mathbf{n},$$

where β is the impedance. Expressed only versus the displacement, the liner boundary conditions reads:

$$c_0^2\,\mathrm{div}(\rho_0\mathbf{u}) = i\omega\beta\,\mathbf{u}\cdot\mathbf{n}.$$

Note that it does not depend on the grazing flow. In particular the velocity is not required to vanish on the lined wall. Such boundary condition is very easy to take into account in the Galbrun model (a numerical example comparing situations with and without impedance on an obstacle is presented in Figure 6) and it leads to a very simple term in a variational form: multiplying Eq. (6) by $\mathbf{u}$ and integrating by parts makes appear the boundary term:

$$\int_{\Gamma_\infty} c_0^2\,\mathrm{div}(\rho_0\mathbf{u})(\bar{\mathbf{u}}\cdot\mathbf{n})d\gamma = i\omega\int_{\Gamma_\infty}\beta|\mathbf{u}\cdot\mathbf{n}|^2 d\gamma.$$

Note that the Myers boundary condition is not convenient to express versus the Euler unknowns. Indeed the link between the normal components of the velocity and of the displacement [22]:

$$\mathbf{v}\cdot\mathbf{n} = \left[\frac{D}{Dt} - \mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\mathbf{v}_0\right](\mathbf{u}\cdot\mathbf{n}),$$

leads to the condition (in the case of a constant impedance to simplify the presentation):

$$\left[\frac{D}{Dt} - \mathbf{n}\cdot(\mathbf{n}\cdot\nabla)\mathbf{v}_0\right]p = -i\omega\beta\,\mathbf{v}\cdot\mathbf{n}.$$

This expression does not appear explicitely in Euler's equations and note also that the gradient of the grazing flow is involved which was not the case with the Galbrun model.

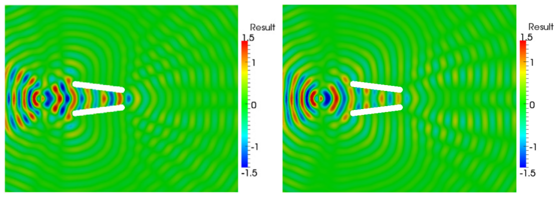

Figure 6: Effect of the impedance value on the acoustic field: rigid case (left) and impedance case (right)

5.3.2 Fluid-structure coupling

It is also convenient to consider a fluid-structure coupling problem in the Galbrun framework. The Galbrun model is well-suited to couple a compressible fluid in flow to an elastic medium since the natural unknown in elasticity is the displacement, as for Galbrun's equation.

Let us consider the interface Γ between a fluid in flow and an elastic solid. The displacement is also noted $\mathbf{u}$ in the solid, characterized by the stress tensor $\sigma(\mathbf{u})$. On Γ the coupling conditions are:

$$\begin{aligned} \sigma(\mathbf{u})\cdot\mathbf{n} &= -p\mathbf{n} = c_0^2\,\mathrm{div}(\rho_0\mathbf{u})\mathbf{n}, \\ [\mathbf{u}\cdot\mathbf{n}] &= 0, \end{aligned}$$

where [] designs the jump accross Γ and $\mathbf{n}$ is a normal vector to Γ. These boundary conditions are natural in the sense that when writing the variational formulation of the coupled Galbrun/elasticity equations the boundary term on Γ vanishes:

$$\int_\Gamma c_0^2\,\mathrm{div}(\rho_0\mathbf{u})(\bar{\mathbf{u}}\cdot\mathbf{n})d\gamma - \int_\Gamma(\sigma(\mathbf{u})\cdot\mathbf{n})(\bar{\mathbf{u}}\cdot\mathbf{n})d\gamma = 0.$$

Acknowledgments

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