

Correction of computational artifacts in numerical solving of diffusion equation for room acoustics

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In the paper a numerical algorithm for solving of diffusion equation in application for room acoustics is presented. The 1D, and 3D models are considered. It has been observed that using simple FTCS algorithm for solving the diffusion equation, an additional energy is generated when the acoustic wave is reflected from the walls. The energy is doubled when the wave reflects from the surface, it is multiplied by 4 when the wave achieves an edge, and it is multiplied by 8 for reflection from the corner. This numerical phenomenon influences energy balance in model and the calculated reverberation times. In 3D model the reverberation time can be inconsistent with Sabine and Eyring formulas, especially for room build with small number of calculation nodes. A heuristic modification of algorithm is proposed. Using this modified algorithm the additional energy is not generated. Reverberation times calculated from modified model are compared with EASE acoustical simulation software, for different room shapes, and room locations.

1 Introduction

Diffusion process in acoustic field prediction is considering from over decade. Diffusion sound field model was developed on the basis of the hypotheses that sound field is uniform in the field, and the sound energy flow is the same in all directions [1]. Sound diffusion theory model was used initially only for a simple shaped rooms, and it took into account only average sound absorption. Simple cases was investigated first: infinite space, long, half infinite corridors [2], or street canyons [3], with small absorption coefficient. After that model was enriched with the boundary conditions, which gave possibility to introduce different absorption coefficients [4], partly with high absorption areas. Measurements in acoustically coupled rooms [5], and comparison with geometrical models were made to diffusion approach accuracy verification. Then Jing and Xiang [6] proposed to include into diffusion equation based model, Eyring's absorption formula, in order to obtain better results in rooms with higher absorption. Despite that diffusion process is closer to scattering nature, next papers improve model range, and give possibilities to use it in rooms with specular reflections [7, 10]. Diffusion equation was also used for calculation of sound transmission through partition wall [8]. In the same time atmospheric attenuation was introduced into diffusion model [9]. Later on, the finite difference method has been used for solving the acoustical diffusion equation [11],[12] which gives possibility to compute acoustical energy in iteration process. This approach as well as problems associated with it will be discussed in this paper.

2 Theory

Simple diffusion equation Eq. (2) after assumptions that for small time changes, local particle-energy flux $J(\mathbf{r},t)$ at a position \mathbf{r} and time t can be approximated with local gradient of the particle density $w(\mathbf{r},t)$ Eq.(3), and when a diffusion coefficient D is constant, can be written as:

$$\frac{\partial w(\boldsymbol{r},t)}{\partial t} - D\nabla^2 w(\boldsymbol{r},t) = 0.$$
(1)

$$\frac{\partial w(\mathbf{r},t)}{\partial t} + \nabla \cdot J = 0.$$
 (2)

$$J(\boldsymbol{r},t) = -D\nabla \big(w(\boldsymbol{r},t)\big) . \tag{3}$$

Introducing the term σ responsible for energy absorption at the room boundaries, Eq. (1) takes the form:

$$\frac{\partial w(\mathbf{r},t)}{\partial t} - D\nabla^2 w(\mathbf{r},t) + \sigma w(\mathbf{r},t) = 0.$$
(4)

Diffusion coefficient *D* is representation of room shape through mean free path λ . Absorption coefficient σ includes mean room absorption coefficient $\bar{\alpha}$, *c* which is speed of sound (particle's speed), volume and the room surface respectively: *V*, *S*.

$$D = \frac{\lambda c}{3} = \frac{4Vc}{3S},\tag{5}$$

$$\sigma = \frac{\bar{\alpha}c}{\lambda} = \frac{\bar{\alpha}cS}{4V},\tag{6}$$

$$\lambda = \frac{4V}{s}.$$
 (7)

Boundary conditions are necessary in the model, because of walls limiting the room. Introducing the source to Eq.(4) we get the equation in the form like in Eq.(8) and boundary condition in the form like in Eq.(9):

$$\frac{\partial w(\boldsymbol{r},t)}{\partial t} - D\nabla^2 w(\boldsymbol{r},t) + \sigma w(\boldsymbol{r},t) = q(\boldsymbol{r},t), \qquad (8)$$

$$\frac{\partial w}{\partial \mathbf{n}} = 0, \tag{9}$$

where $q(\mathbf{r},t)$ is a strength of sound source, define number of phonons in time and volume unit in point r, and **n** is the outgoing normal vector of the wall surface. Neumann boundary condition Eq.(9) don't allow energy to leave the model. Moving absorption factor from volume to surface domain the equation and boundary condition can be written as:

$$\frac{\partial w(\mathbf{r},t)}{\partial t} - D\nabla^2 w(\mathbf{r},t) = q(\mathbf{r},t), \qquad (10)$$

$$J(\mathbf{r},t) = -D\frac{\partial w(\mathbf{r},t)}{\partial n} = \frac{c\alpha}{4}w(\mathbf{r},t), \qquad (11)$$

Since article takes into account more general model behaviour, for more details please refer to [4].

3 Solving method

To solve diffusion equation, it was decided to use finite difference method. It gives possibility to solve parabolic equations in simple iteration process. Using differential quotients, and FTCS (Forward Time, Central Space) method, simple parabolic equation of type:

$$\frac{\partial W}{\partial t} = D \frac{\partial^2 W}{\partial x^2},\tag{12}$$

can be written in the discrete form:

$$\frac{W^{t+1}-W^{t}}{\Delta t} = D \frac{W_{l+1}^{t}-2W^{t}+W_{l-1}^{t}}{\Delta x^{2}},$$
(13)

or after simple transformation:

$$W_i^{t+1} = W_i^t + \delta(W_{i+1}^t - 2W_i^t + W_{i-1}^t), \quad (14)$$

$$\delta = D \frac{\Delta t}{\Delta x^2},\tag{15}$$

where:

 Δt – time step Δx – spatial step in direction of x D – diffusion coefficient *i* – node index along x axis

Basing on values of surrounding points (nodes) in the previous time step, it is possible to calculate values in given nodes in present time step. We get so-called explicit scheme of equation, called also FTCS scheme (Forward Time, Central Space). Method is numerical stable for $0<\delta<0.5$ for 1D case what means that for these values, error in next iterations is not growing.

Finally, the diffusion equations (10) and (11), have the finite-difference form:

$$\frac{W_i^{t+1} - W_i^t}{\Delta t} - D \frac{W_{i+1}^{t-2}W_i^t + W_{i-1}^t}{\Delta x^2} = Q_i^t \quad \text{in } V,$$
(16)

$$-D\frac{W^{t}_{i+1} - W^{t}_{i-1}}{2\Delta x} = \frac{c\alpha}{4}W^{t}_{i} \qquad \text{on } S.$$
(17)

Calculations in all points in volume are made on the base of surrounding, neighbouring points. The values on the wall are calculated on the basis of a neighbouring point lying next to the wall and the neighbouring point lying outside the wall. To calculate on the wall node's value, the value of point lying outside the room have be calculated:

$$W^{t}_{i+1} = -\frac{2\Delta x c \alpha}{4D} W^{t}_{i} + W^{t}_{i-1}.$$
 (18)

To eliminate necessity of computing points outside the model, the wall equation can be transformed. By substituting the expressions for the energy in node outside the wall Eq.(18) into volume equation Eq.(16), expression for the value on the wall, eliminating the necessity of beyond the wall values calculation, is determined Eq.(19). Then both equations transformed in order to calculate given energy node value, look:

$$W_i^{t+1} = W_i^t (1 - 2\delta) + \delta(W_{i+1}^t + W_{i-1}^t) + \Delta t Q_i^t \quad \text{in } V,$$
(19)

$$W_i^{t+1} = W_i^t \left(1 - 2\delta - \frac{\Delta t c \alpha}{2\Delta x} \right) + 2\delta W_{i-1}^t \qquad \text{on } S.$$
 (20)

For cases when source is off, the expression $\Delta t Q_i^t$ is removed from Eq.(19).

4 Three-dimensional model

In 3D case the volume of model is represented by net of regularly spaced neighbouring points. To simplify calculation, nodes are spaced regularly in all directions, making a cubic measure grid with spatial step Δx . In the case of three-dimensional space without source Eq.(16) and Eq.(17) take the form:

$$W_{i,j,k}^{t+1} = W_{i,j,k}^{t}(1-6\delta) + \delta \begin{pmatrix} W_{i+1,j,k}^{t} + W_{i-1,j,k}^{t} + W_{i,j+1,k}^{t} + \\ + W_{i,j-1,k}^{t} + W_{i,j,k+1}^{t} + W_{i,j,k-1}^{t} \end{pmatrix}$$

$$-D \frac{W^{t_{i+1}-W^{t_{i-1}}}}{2\Delta x} = \frac{c\alpha}{4} W^{t_{i}} \qquad \text{on } S_i, \quad (22)$$

$$-D\frac{W^{t_{j+1}-W^{t_{j-1}}}}{2\Delta x} = \frac{c\alpha}{4}W^{t_{j}} \qquad \text{on } S_{j}, \qquad (23)$$

$$-D\frac{W^{t_{k+1}-W^{t_{k-1}}}}{2\Delta x} = \frac{c\alpha}{4}W^{t_{k}} \qquad \text{on } S_{k}, \quad (24)$$

where *i*,*j*,*k* are the nodes index respectively along axis *x*,*y*,*z*. For 3D case method is numerical stable when $0 \le \delta \le 0.16(6)$.

Like in 1D case to eliminate necessity of computing points outside the model, the wall equation can be modified. This time there are not only walls like in 1D case, but also edges (intersection of two walls), and corners (intersection of three walls), so it's more complicated, but the way of transformation is analogical to 1D case:

Equations for the edges:

$$\begin{split} W_{i,j,k}^{t+1} &= W_{i,j,k}^{t} \left(1 - 6\delta - \frac{\Delta t c \alpha}{2\Delta x} \right) + \\ &+ \delta \left(2W_{i-1,j,k}^{t} + 2W_{i,j-1,k}^{t} + W_{i,j,k-1}^{t} + W_{i,j,k+1}^{t} \right) & \text{on } S_{ij}, \quad (27) \\ W_{i,j,k}^{t+1} &= W_{i,j,k}^{t} \left(1 - 6\delta - \frac{\Delta t c \alpha}{2\Delta x} \right) + \\ &+ \delta \left(2W_{i-1,j,k}^{t} + W_{i,j-1,k}^{t} + W_{i,j+1,k}^{t} + 2W_{i,j,k-1}^{t} \right) & \text{on } S_{ik}, \quad (28) \\ W_{i,j,k}^{t+1} &= W_{i,j,k}^{t} \left(1 - 6\delta - \frac{\Delta t c \alpha}{2\Delta x} \right) + \\ &+ \delta \left(W_{i-1,j,k}^{t} + W_{i+1,j,k}^{t} + 2W_{i,j-1,k}^{t} + 2W_{i,j,k-1}^{t} \right) & \text{on } S_{jk}. \quad (29) \end{split}$$

And finally equation for corners:

$$W_{i,j,k}^{t+1} = W_{i,j,k}^{t} \left(1 - 6\delta - \frac{\Delta t c \alpha}{2\Delta x}\right) + \delta \left(2W_{i-1,j,k}^{t} + 2W_{i,j-1,k}^{t} + 2W_{i,j,k-1}^{t}\right)$$

on S_{iik} . (30)



Figure 1: Schematic draw of calculation nodes net. With darker color are marked nodes on the basis of the energy of node placed in the centre is calculated.

5 Numerical model modifications

implementation some During model numerical problems appear. First of all it has been noticed that for small absorption coefficients additional energy is generated in model. Additional energy increases with decreasing number of calculation points in model. Below the energy decay for cubic room where wall absorption was $\alpha=0$ is presented. The number of points model was build of, are: 6x6x6 pts, 11x11x11 pts and 21x21x21 pts. The source of energy, equal to 1000 for time t = 0 and 0 for t > 0 was placed in the centre of the room. The values of the spatial and time step were as follows: $\Delta x = 1$, $\Delta t = 0.0001$. It can be seen that the energy increases to values higher than the source value. This behaviour hasn't physical explanation.



Figure 2: Energy distribution in 3D model as a function of model size for values: $\alpha=0$, $\Delta x=1$ m; $\Delta t=0,0001$ s.



Figure 3: Energy distribution in 6 x 6 x 6 pts model as a function of absorption coefficient, for values: $\Delta x=1m$; $\Delta t=0,0001s$.

After observation founded that the energy is doubled when the wave reflects from the surface points, it is multiplied by 4 when the wave achieves an edges, and it is multiplied by 8 for reflection from the corners. To eliminate that phenomenon equations for wall, edge and corner showed in Eg.(24)-Eq.(30) have been modified:

$$\begin{split} W_{i,j,k}^{t+1} &= \left(W_{i,j,k}^{t} \left(1 - 6\delta - \frac{\Delta t c \alpha}{2\Delta x} \right) + \right. \\ &+ \delta \left(W_{i-1,j,k}^{t} + W_{i+1,j,k}^{t} + 2W_{i,j-1,k}^{t} + W_{i,j,k-1}^{t} + W_{i,j,k+1}^{t} \right) \right) \cdot \frac{1}{2} + \\ &+ \frac{1}{2} W_{i,j,k}^{t} \qquad \text{on } S_{j}, \quad (25) \\ W_{i,j,k}^{t+1} &= \left(W_{i,j,k}^{t} \left(1 - 6\delta - \frac{\Delta t c \alpha}{2\Delta x} \right) + \right. \\ &+ \delta \left(W_{i-1,j,k}^{t} + W_{i+1,j,k}^{t} + W_{i,j-1,k}^{t} + W_{i,j+1,k}^{t} + 2W_{i,j,k-1}^{t} \right) \right) \cdot \frac{1}{2} + \\ &+ \frac{1}{2} W_{i,j,k}^{t} \qquad \text{on } S_{j}, \quad (26) \end{split}$$

Equations for the edges:

And finally equation for corners:

After modification showed above energy doesn't rise over the source value.



Figure 4: Energy distribution in 6 x 6 x 6 pts model as a function of absorption coefficient, for values: $\Delta x=1m$; $\Delta t=0,0001$ s, after modification.

Model with the modification doesn't generate additional energy which is showed in Figure 1, but unfortunately new numerical problem appears. This time in modified model energy absorption is faster when model is build with a larger number of calculation nodes, although the geometric dimensions are preserved. It was noticed during reverberation time calculations. It was obvious that earlier modification influenced on model behavior and destroyed its capability to compensate number of calculation nodes changes. To save the usefulness of the model, the model had to be equipped with tool to eliminate the previously described behavior. After comparisons of the results taken from models with different nodes morphology, discovered that problem lies in number of points on the walls to edges to corner ratio. When model is growing, or decreasing ratio of points in: volume, on the walls, edges, and corner is changing, despite that geometrical dimensions of the model are the same. Because modification changed behavior of the model on the walls, edges and corners, it implicated changes in energy balance during ratio changing. Because walls, edges and corners are responsible for energy absorption, to fix this problem new coefficient is introduced to surface equations:

$$K = \frac{8(V_N + W_N + E_N + C_N)}{8V_N + 4W_N + 2E_N + C_N},$$
(31)

where:

 V_N – number of nodes in volume,

 W_N – number of nodes on the walls,

 E_N – number of nodes in the edges,

 C_N -number of nodes in the corners (eight in a cubic room). Below is the example of equation with coefficient K,

made on the base of Eq.(24). Coefficient is placed in factor which is responsible for energy absorption.

In other surface equations (walls, edges, corners), coefficient is placed in the same place. Coefficient *K* takes the value 8, for the smallest possible model: $2 \times 2 \times 2$ pts, and heading to unity for bigger models.





Like in the previous figures the source of energy, equal to 1000 for time t = 0 and 0 for t>0 is placed in the centre of the room. This time Figure 5 shows energy decays for cubic room 10x10x10m built from 3x3x3pts, and 11x11x11pts, modelled using modified and unmodified equations, for spatial step $\Delta x=1$ m (11pts model) and $\Delta x=5$ m (3 pts model) and time step $\Delta t=0,0001$ s. As it can be seen on Figure 5 modifications not only keep energy below source value but also cause that energy decays for model build from different calculation points number are practically identical, while energy decays of unmodified model are different. Modifications caused, that model behaves more stable and they give possibility to calculate acoustical energy even when built with just few calculation nodes.

6 Reverberation time calculations

In this section results from RT60 calculations using modified diffusion model and acoustic simulation software EASE 4.3 with Aura module are presented. "Based on the CAESAR algorithms developed by Aachen University (RWTH), AURA allows the calculation of all key room acoustical parameters defined in ISO3382, the International Standard on Room Acoustic Measurements" [13].



Figure 6: Location of source and recivers during RT60 calculations, on the example of long room: 6 x 22 x 6 m.

Calculations have been done for three receivers and one source location. During simulation in EASE software all surfaces in model had scattering coefficient equal to unity, to model diffusion sound filed to allow on comparison EASE results with diffusion model results. Time of simulation window in EASE had at least $\frac{3}{4}$ of simulated room RT60. Number of particles generated in simulation for cubic, long and flat room were respectively: 72000, 57000,224000. Before simulation with these number of particles, preliminary simulation for larger number of particles were made. Simulation results did not differ. Calculation for diffusion model have been done when absorption coefficient α in surfaces equation have been changed with factor known form classical Eyring formula [6]: $-\log(1-\alpha)$.

For both diffusion and EASE model, reverberation times were calculated as a time of energy decay from -5 to -35dB in given point, and multiplied by two.

Table 1: RT60 results for cubic room: 10 x 10 x 10 m.

Cubic room: 10 x 10 x 10 m (11 x 11 x 11 pts)							
1 - corner: 1x1x1 m; 2 - wall: 5x1x5 m; 3 - middle: 5x4x5 m							
α	Eyring RT60 [s]	Modified Diff.			EASE		
		RT60 in point [s]			RT60 in point [s]		
		1	2	3	1	2	3
0,1	2,53	2,58	2,58	2,58	2,57	2,58	2,58
0,2	1,20	1,24	1,24	1,23	1,25	1,25	1,25
0,3	0,75	0,79	0,79	0,78	0,79	0,80	0,80
0,4	0,52	0,56	0,56	0,55	0,57	0,57	0,58
0,5	0,38	0,43	0,42	0,41	0,43	0,43	0,44
0,6	0,29	0,33	0,33	0,32	0,33	0,33	0,33
0,7	0,22	0,26	0,26	0,25	0,26	0,26	0,28
0,8	0,17	0,21	0,21	0,19	0,20	0,21	0,22
0,9	0,12	0,16	0,16	0,15	0,15	0,16	0,19
0,99	0,06	0,11	0,11	0,10	0,13	0,13	0,27

Reverberation times for EASE and diffusion model are in very good agreement. The only difference between these two models are in calculation in close proximity to the source. But even then results obtained from diffusion model are closer to RT60 taken from classical Eyring formula which also is presented in tables.

Table 2: RT60 results for long room: 6 x 22 x 6 m.

Long room: 6 x 22 x 6 m (7 x 23 x 7 pts)									
Calculation in point:									
1 - corner: 1x1x1 m; 2 - wall: 3x1x3 m; 3 - middle: 3x10x3 m									
α	Eyring RT60 [s]	Modified Diff.			EASE				
		RT60 in point [s]			RT60 in point [s]				
		1	2	3	1	2	3		
0,1	2,00	2,04	2,04	2,01	2,04	2,04	2,05		
0,2	0,95	0,98	0,98	0,94	1,00	1,00	0,99		
0,3	0,59	0,63	0,63	0,59	0,65	0,64	0,62		
0,4	0,41	0,45	0,45	0,41	0,45	0,46	0,45		
0,5	0,30	0,34	0,34	0,31	0,35	0,35	0,33		
0,6	0,23	0,27	0,27	0,23	0,29	0,28	0,27		
0,7	0,18	0,21	0,21	0,18	0,22	0,23	0,20		
0,8	0,13	0,17	0,17	0,14	0,18	0,21	0,18		
0,9	0,09	0,13	0,13	0,11	0,16	0,13	0,12		
0,99	0,05	0,09	0,09	0,07	0,07	0,07	0,11		

Table 3: RT60 results for flat room: 25 x 25 x 5 m.

Flat room: 25 x 25 x 5 m (51 x 51 x 11 pts) Calculation in point: 1 - corner: 1x1x1 m; 2 - wall: 12,5 x 1 x 12,5 m; 3 - middle: 12.5 x 11.5 x 12.5 m								
α	Eyring RT60 [s]	Modified Diff. RT60 in point			EASE RT60 in point			
		1	2	3	1	2	3	
0,1	2,71	2,77	2,77	2,73	2,78	2,78	2,76	
0,2	1,28	1,33	1,33	1,25	1,35	1,36	1,34	
0,3	0,80	0,85	0,85	0,77	0,88	0,88	0,84	
0,4	0,56	0,61	0,61	0,53	0,66	0,65	0,59	
0,5	0,41	0,46	0,46	0,40	0,49	0,51	0,44	
0,6	0,31	0,36	0,36	0,30	0,40	0,41	0,35	
0,7	0,24	0,29	0,29	0,24	0,36	0,32	0,28	
0,8	0,18	0,23	0,23	0,18	0,33	0,33	0,26	
0,9	0,12	0,18	0,17	0,14	0,28	0,22	0,20	
0,99	0,06	0,11	0,11	0,08	0,25	0,25	0,09	

7 Conclusion

In the paper a numerical algorithm for solving of diffusion equation using finite different method was presented. Algorithm gives possibility to solve diffusion equation in iteration process. Introduced modifications stabilized energy balance and give possibility to make calculation even for model built with small number of points. Results compared to EASE 4.3 with Aura module software shows good agreement in RT60 calculations.

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