Analysis of the nonlinear effect of the capo bar-string interaction in grand piano

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The treble strings of grand pianos usually terminate at a capo bar, which is situated above the strings. The apex of a V-shaped section of the capo bar defines the end of the speaking length of the strings. Accurate modeling of the piano string - capo bar interaction requires that the curvature of V-shaped section be recognized and included. The model that realizes this with numerical calculation is described, and results relating the effect of the capo bar are presented. The finite curvature of the capo bar at the point of the piano string termination causes the nonlinear vibrations of the string. As a result, the spectrum of the string vibrations depends not only on the amplitude of the hammer impact, but the spectral structure changes continuously as time passes, even when the hammer has left the string and it vibrates freely after that.

1 Introduction

This paper presents a physics-based model for simulation of vibrations of piano string, which at one end has the ideal rigid support, and another end is terminated at a capo bar. Investigation of the boundary condition of vibrating string is a very important problem in musical acoustics. It is well known that the fundamental frequency of piano string is strictly determined by the type of the string termination.

The types of the string support in the piano are different for the bass and treble notes. All the far ends of the piano strings are terminated on the bass and treble bridges, which are rather complicated resonant systems. The nearest ends of the bass and long treble strings begin from the agraffe that can be considered as an absolutely rigid clamp termination. However the most part of the treble strings starts from the capo bar - the rigid edge of the cast iron frame. These strings are bent around the capo bar, and their vibration tone depends on the curvature of the capo bar V-shaped section. The same type of the string support can be seen also on the guitar and some other musical string instruments.

Usually the changing of the tone caused by the curvature of the string support is negligible, but there is a family of Japanese plucked stringed instruments (biwa and shamisen), which sounding is strictly determined by the string termination [1, 2]. These lutes are equipped with a mechanism called "sawari" (touch). The sawari is a contact surface of very limited size, located at the nut-side end of the string, to which the string touches repeatedly, producing a unique timbre of the instrumental tone called the sawari tone.

The aim of this paper is to show the influence of the contact nonlinearity on the spectral structure of the piano string vibration. This can be divided into two stages. Firstly, mathematical modeling of the hammer-string interaction enables prediction of the the piano string motion [3, 4]. Secondly, this knowledge is used for appropriate simulation of interaction of the vibrating string with a capo bar.

The numerical simulation of the hammer-string interaction is based on the physical models of a piano hammer described in [5, 6, 7]. These models are based on the assumption that the woollen hammer felt is a microstructural material possessing history-dependent properties. The elastic and hereditary parameters of piano hammers were obtained experimentally using a special piano hammer testing device that was developed and built in the Institute of Cybernetics at Tallinn University of Technology [7].

In this paper a number of simplifying assumptions regarding the string and string supports are introduced. Thus, the piano string is assumed to be an ideal flexible string, but the coupling of strings at the end supports is neglected, and the bridge motion is also ignored. We also assume that the right string termination (bridge) is the ideal rigid support. The left string termination (capo bar) is considered here as a rigid but not ideal support, because we take into account the curvature of its V-shaped section. Nevertheless, it is hoped that the application of the proposed model may clarify the physics of vibration of the string with nonlinear support.

2 String excitation and basic formulae

In this paper it is assumed that the piano string is an ideal (flexible) string. The displacement $y(x, t)$ of such a string obeys the simple wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}. \tag{1}$$

As in [3], we have the system of equations describing the hammer-string interaction

$$\frac{dz}{dt} = -\frac{2T}{cm} g(t) + V, \tag{2}$$

$$\frac{dg}{dt} = \frac{c}{2T} F(t), \tag{3}$$

where $g(t)$ is the outgoing wave created by the hammer strike at the contact point $x = l$, c is the speed of a transverse nondispersive wave traveling along the string; $F(t)$ is the acting force, T is the string tension; m, $z(t)$, and V are the hammer mass, the hammer displacement, and the hammer velocity, respectively. The hammer felt compression is determined by $u(t) = z(t) - y(l, t)$. Function $y(l, t)$ describes the string transverse displacement at the contact point $x = l$, and is given by [4]

$$y(l, t) = g(t) + 2\sum_{i=1}^{\infty} g\left(t - \frac{2iL}{c}\right) - \sum_{i=0}^{\infty} g\left[t - \frac{2(i+a)L}{c}\right] - \sum_{i=0}^{\infty} g\left[t - \frac{2(i+b)L}{c}\right]. \tag{4}$$

It is assumed that the string of length L extends from $x = 0$ on the left to $x = L$. Parameter $a = l/L$ is the fractional length of the string to striking point, and $b = 1 - a$. Parameter a determines the actual distance l of the striking point from nearest string end. The initial conditions at the moment when the hammer first contacts the string, are taken as $g(0) = z(0) = 0$, and $dz(0)/dt = V$.

The physical interpretation of Eq. (4) is very simple. It describes the deflection of the string at the contact point that is determined by the traveling waves moving in both directions along the string and reflecting back from the string supports. Here the index of summation i simply denotes the number of rereflections.

The experimental testing of piano hammers demonstrates that all hammers have a hysteretic type of force-compression characteristics. A main feature of hammers is that the slope of the force-compression characteristics is strongly dependent on the rate of loading. It was shown that nonlinear hysteretic models can describe the dynamic behavior of the hammer felt [5, 6, 7]. These models are based on assumption that the hammer felt made of wool is a microstructural material possessing history-dependent properties. Such a physical substance is called either a hereditary material or a material with memory.

According to a three-parameter hereditary model of the hammer presented in [6], the nonlinear force $F(t)$ exerted by the hammer is related to the felt compression $u(t)$ by the following expression

$$F(u(t)) = Q_0 \left[u^p + \alpha \frac{d(u^p)}{dt} \right]. \qquad (5)$$

Here the parameter Q_0 is the static hammer stiffness; p is compliance nonlinearity exponent, and α is the retarded time parameter.

The hammer-string interaction is simulated using the basic formulae presented above. For numerical simulation of the piano string with nonlinear support we chose here the note number $n = 82$ (tone $F\sharp7$, frequency $f = 2960$ Hz). The string parameters are the following: the string length $L = 71$ mm; the actual distance of the striking point from nearest string end $l = 3.5$ mm; the string tension $T = 742$ N.

The continuous variations in hammer parameters across the compass of the piano were obtained experimentally by measuring a whole hammer set of recently produced unvoiced *Abel* hammers. The result of those experiments is presented in [6, 7]. A best match to the whole set of hammers $1 \leqslant n \leqslant 88$ was approximated using

$$\begin{aligned} Q_0 &= 183 \exp(0.045n), \\ p &= 3.7 + 0.015n, \\ \alpha &= 259.5 + 0.58n + 6.6 \cdot 10^{-2} n^2 \\ &- 1.25 \cdot 10^{-3} n^3 + 1.172 \cdot 10^{-5} n^4, \end{aligned} \qquad (6)$$

Here the dimension of parameter α is [ms], and the dimension of Q_0 is [N/mm^p].

The hammer masses of this set were approximated by

$$m = 11.074 - 0.074n + 10^{-4} n^2, \quad 1 \leqslant n \leqslant 88. \qquad (7)$$

The mass of hammer $n = 1$ ($A0$) is 11.0 g, and the mass of hammer $n = 88$ ($C8$) 5.3 g. For the hammer number $n = 82$ we use the following parameters: static stiffness $Q_0 = 7328$ N/mm^p; nonlinearity exponent $p = 4.93$; hereditary parameter $\alpha = 0.496$ ms.

The V-shaped section of the capo bar has approximately a parabolic form, and it is described here by the function $W(x) = (2R)^{-1}x^2$, where R is the radius of the capo bar curvature at $x = 0$.

The simulation of the hammer-string interaction was provided by solving the system of equations Eq. (2), Eq. (3) for initial hammer velocity $V = 2$ m/s, and for one string per note. For this purpose the acting mass of a hammer is defined as being the total hammer mass divided by the number of strings per note $n = 3$. Thus the hammer mass used is $m = 1.9$ g.

As a result of simulation we can find the history of the acting force $F(t)$ and the time dependence of the outgoing wave $g(t)$ created by the hammer strike.

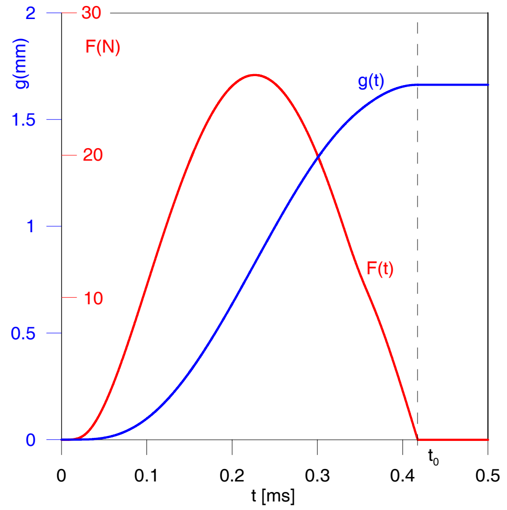

Figure 1: Force history and g function computed for tone $F\sharp7$ ($f = 2960$ Hz). The dashed line defines duration of contact t_0.

These functions are presented in Figure 1. The vertical dashed line defines duration of contact t_0 between the hammer and the string. When the hammer has lost the contact with the string the acting force $F(t) = 0$. Therefore, according to Eq. (3), the outgoing wave $g(t) = const$ for any time $t > t_0$.

3 Capo bar-string interaction

The proposed model of the capo bar-string interaction is based on the knowledge of the outgoing wave function $g(t)$ created by the hammer strike. It is evident that Eq. (1) may be satisfied by combination of simple nondispersive waves $g_1(t - x/c)$ and $g_2(t + x/c)$ moving in either directions along the string from the point $x = l$ where the string makes contact with the hammer. At this point $g_1(t) = g_2(t) = g(t)$. These two waves g_1 and g_2 are simply translation of outgoing wave $g(t)$ from the point $x = l$ to the other segment of the string, and their amplitudes are always positive, because $g(t) > 0$.

These two waves are being reflected from each end of the string. The wave g_1 moving to the right creates the wave $g_4(t + x/c)$ moving to the left. At the right end of the string $x = L$ the reflected wave $g_4(t) = -g_1(t)$, because this string termination is ideal rigid. The wave g_2 moving to the left also creates the wave $g_3(t - x/c)$ moving to the right. But at the left end of the string the reflection is more complicated.

In this paper we assume that the reflecting wave $g_3(t - x/c)$ moving to the right appears only at the point $x = x_*$, where the amplitude of the sting deflection $y(x_*, t) = W(x_*)$. The position of this point x_* is determined by the V-shape form of the capo bar. At the point of reflection $x = x_*$ we have $g_3(t) = -g_2(t)$, because we assume that capo bar is also ideal rigid. The amplitudes of reflected waves $g_3(t - x/c)$ and $g_4(t + x/c)$ are always negative. The scheme of waves propagation along the sting is shown in Figure 2.

The physical interpretation of the functions g_3 and g_4 shows what we should use for their values: they exist only because the outgoing wave g at some earlier time has been reflected from the string ends. According to our model, the string deflection $y(x, t)$ (shown in Figure 2 by red line) at any point x and at any time t is simply the resulting sum of wave-

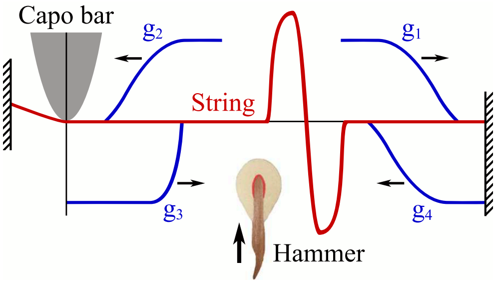

Figure 2: Scheme of capo bar-string interaction. Functions g are the traveling waves.

forms g moving in both directions:

$$y(x,t) = g_1(t-x/c)+g_2(t+x/c)+g_3(t-x/c)+g_4(t+x/c). \quad (8)$$

A computing method that realizes the calculation of the string deflection determined by Eq. (8) is based on a digital delay-line procedure. The numerical application of this method is best explained by Hall in Appendix A [3].

4 Results and analysis

With the model described in Section 3 it is possible to compute the string vibrations in any point of the string. Therefore, we can simulate the interaction of the string with a capo bar, and investigate the effect of contact nonlinearity on the spectral structure of the piano string vibration.

The influence of the contact nonlinearity on the process of wave propagation along the string is significant. Figure 3 shows reflection of the traveling wave g_2 from the capo bar. With each reflection the front of reflecting wave g_3 becomes steeper, and after only fifteen reflections the wave breaks.

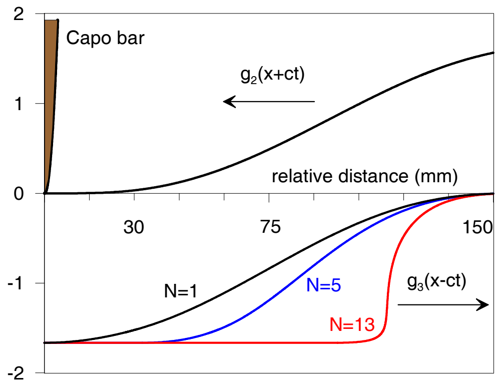

Figure 3: Distortion of the form of reflecting wave g_3 after several number of rereflecions $N = 1; 5; 13$, shown in different colors.

It means that our model may be used for description of process of the string vibration only for the period of 10-13 vibrations. This time interval for tone $F\sharp7$ (f = 2960 Hz) may be estimated as 5 ms approximately. In reality the high frequency patials lose energy very quickly. The present model does not take into account the frequency-depending damping of the high frequency partials.

The nonlinear effect caused by the capo bar-string interaction was studied through spectrograms. Figure 4 shows the spectrograms of simulated tone $F\sharp7$ (f=2960 Hz) computed for constant value of hammer striking velocity V= 2 m/s, and for various radius of the capo bar curvature R.

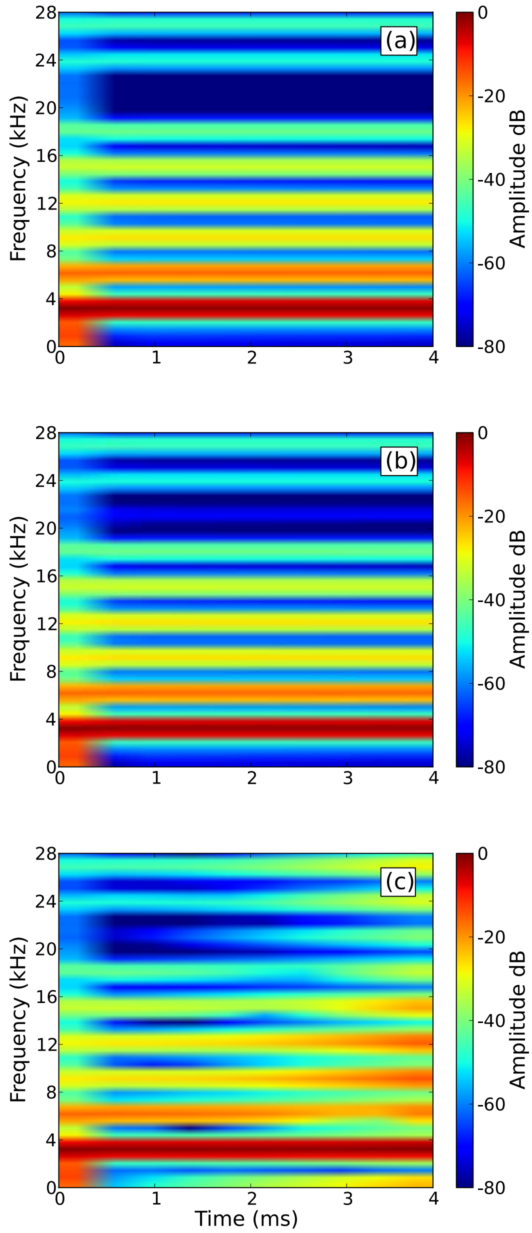

Figure 4: Spectrograms of simulated tone $F\sharp7$ (f=2960 Hz) calculated for constant value of hammer striking velocity V= 2 m/s, and varying radius of the capo bar curvature R. (a) R = 0, (b) R = 1 mm, (c) R = 2 mm.

Figure 5 shows the spectrograms of the same simulated tone $F\sharp7$ (f=2960 Hz) computed for constant value of the capo bar curvature R = 1mm, and for various velocities V of the hammer.

All spectrograms were computed using a Chebyshev window of length 1.7 ms with a 80% overlap, and at the point of the string, which is located at the distance $x = 1/7L$ measured from agraffe. Therefore, in Figure 4(a), and also in Figure 5(a) that present vibrations of the string with ideal supports, the 7th harmonic is absent.

Figure 4(b) demonstrates that with increasing of radius of the capo bar curvature the 7th harmonic is appeared. In Figures 4(c) and 5(c) the amplitude of this harmonic is significant.

As well in Figures 4(c) and 5(c), which implies that the model presents well the nonlinear effect of the capo bar influence on the spectra of the string vibrations, the dynamic

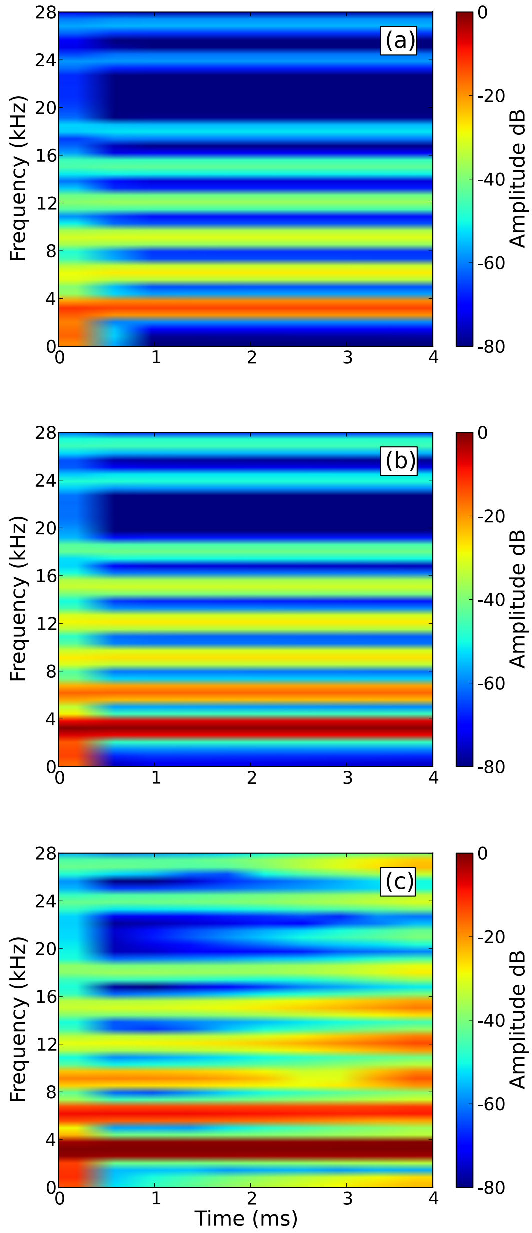

Figure 5: Spectrograms of simulated tone $F\sharp7$ (f=2960 Hz) calculated for constant value of the capo bar curvature R = 1mm, and varying the hammer striking velocity V. (a) V = 1 m/s, (b) V = 2 m/s, (c) V = 3 m/s.

level of harmonics changes continuously as time passes, even when the hammer has left the string. This is visible as "spreading" in the vicinity of the horizontal lines of third, fourth and fifth harmonics in Figures 4(c) and 5(c).

Figures 4(c) and 5(c) that present the spectrograms of the simulated tone with nonlinear effect of the capo bar influence show that this effect gains energy of high partials. Thus, with the suggested model it is possible to imitate the energy transfer from the lower partials to the higher partials,

5 Conclusions

This paper presented a method for simulating the nonlinear effects of string vibrations caused by the interaction between the capo bar and the string. One respect in which this model is still idealized is its assumption about a very simple string boundary condition at the piano bridge.

It is found that the high frequency oscillations that do not exist initially grow up eventually, and their appearance depends on the curvature of the of V-shape of the capo bar. It is shown that the power spectrum of the string vibration is enriched by spectral components up to very large numbers, and essentially reshapes with increasing of the amplitude of the initial wave exited by piano hammer.

It is revealed that even the small variation of the edge curvature significantly influenced the amplitude of the patials. For this reason the manufacturers of grand pianos should reproduce a V-shape form of capo bar very accurately, and carefully process the surface of its edge.

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