Acoustic analogy in an annular duct with swirling mean flow

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The present study aims at developing an acoustic formulation for the sound generated by the interaction of solid surfaces (such as blades) with an unsteady flow in an annular duct with swirl. Indeed, the mean flow in between the rotor and the stator of the fan or of a compressor stage is highly swirling. As a result, in order to properly predict the rotor self-noise radiated downstream of the rotor or the rotor-stator interaction noise radiated upstream of the stator, the swirling mean flow effect must be accounted for, either in the source terms or in the differential operator in an acoustic analogy. The proposed approach here, is to develop an acoustic analogy with an operator accounting for the swirl. It can be seen as an extension of Goldstein formulation in uniform mean flow (Aeroacoustics, 1976). The Navier-Stokes equations are first recast to obtain the differential operator and the associated equivalent noise sources in space and time. Then, the Green's function tailored to the rigid annular duct with swirl is derived in the frequency domain. Finally, the formulation to be used in the fan noise context is outlined.

1 Introduction

The mean flow in between the rotor and the stator of the fan or of a compressor stage is highly swirling. Several studies [1, 2, 3] have shown that the swirl modifies the number of acoustic modes in the duct, their radial profile and alters the incident disturbance in rotor-stator interaction. The present study is a part of an ongoing work dedicated to account for the swirling mean flow effect on rotor-stator fan noise prediction. In particular, it aims at developing an acoustic analogy in an annular duct which can be written in a very similar form as was previously done with uniform mean flow (e.g. [4]), namely, exhibiting the product of the pressure distribution on the blades (or pressure jump) with an operator acting on a tailored Green's function.

2 Navier-Stokes equations: linearised inhomogeneous Euler equations

Let us consider an infinite cylindrical annulus, $h \leqslant r \leqslant 1$, in cylindrical polar coordinates (r, θ, x_d), with hard impermeable walls at $r = h$ and $r = 1$. Throughout, lengths are made non-dimensional by the outer radius of the duct, densities by the mean flow density at $r = 1$ and velocities by the mean sound speed at $r = 1$. Let $\mathbf{u}_{to} = (u_{to}, v_{to}, w_{to})$, ρ_{to}, and p_{to} be the total variables (velocity vector, density and pressure); the mean flow be subsonic inviscid, of velocity components:

$$\mathbf{U} = (U_r, U_\theta, U_{xd}) = (0, U_\theta(r), U_{xd}(r)) \ , \tag{1}$$

density $\rho_0(r)$ and pressure $P_0(r)$; and $\mathbf{u} = (u, v, w)$, ρ and p be the associated fluctuating variables of the perturbation. That is to say, $\mathbf{u}_{to} = \mathbf{U} + \mathbf{u}$, $\rho_{to} = \rho_0 + \rho$ and $p_{to} = P_0 + p$. The Navier-Stokes equations in cylindrical coordinates can be exactly written as a linear operator acting on the perturbations subject to an inhomogeneous right hand side including all non-linear effects:

$$\frac{1}{c_0^2}\frac{\mathrm{D}_0 p}{\mathrm{D}t} + u\frac{\mathrm{d}\rho_0}{\mathrm{d}r} + \rho_0 \operatorname{div}\mathbf{u} = -\operatorname{div}(\rho\mathbf{u}) + \frac{\mathrm{D}_0 Z}{\mathrm{D}t} = S_\rho\,, \tag{2}$$

$$\rho_0\left[\frac{\mathrm{D}_0 v}{\mathrm{D}t} + \frac{u}{r}\frac{\mathrm{d}(r\,U_\theta)}{\mathrm{d}r}\right] + \frac{1}{r}\frac{\partial p}{\partial\theta} = S_\theta\,, \tag{3}$$

$$\rho_0\left[\frac{\mathrm{D}_0 u}{\mathrm{D}t} - 2\frac{U_\theta}{r}v\right] + \frac{\partial p}{\partial r} - \frac{U_\theta^2}{r\,c_0^2}p = S_r\,, \tag{4}$$

$$\rho_0\left[\frac{\mathrm{D}_0 w}{\mathrm{D}t} + u\frac{\mathrm{d}U_{xd}}{\mathrm{d}r}\right] + \frac{\partial p}{\partial x_d} = S_x\,, \tag{5}$$

with the energy equation (not detailed), where

$$\frac{\mathrm{D}_0}{\mathrm{D}t} = \frac{\partial}{\partial t} + U_{xd}\frac{\partial}{\partial x_d} + \frac{U_\theta}{r}\frac{\partial}{\partial\theta} \tag{6}$$

is the linear convective derivative operator with t the time,

$$S_\rho = -\operatorname{div}(\rho\mathbf{u}) + \frac{\mathrm{D}_0 Z}{\mathrm{D}t}\,, \tag{7}$$

with

$$Z = (p - c_0^2\rho)/c_0^2\,, \tag{8}$$

and the vector $\mathbf{S} = (S_r, S_\theta, S_x)$ is defined by:

$$\mathbf{S} = \boldsymbol{\nabla}\cdot\overline{\overline{\boldsymbol{\tau}}} - \rho_{to}\,(\mathbf{u}\cdot\boldsymbol{\nabla})\,\mathbf{u} - \rho\frac{\mathrm{D}_0\mathbf{u}}{\mathrm{D}t_0} - \rho\mathcal{H} - \frac{U_\theta^2}{r_0}Z\mathbf{e_r}\,, \tag{9}$$

where $\overline{\overline{\boldsymbol{\tau}}}$ is the viscous stress tensor and

$$\mathcal{H} = -2\frac{U_\theta}{r_0}v\,\mathbf{e_r} + \frac{u}{r_0}\frac{\mathrm{d}\,(r_0 U_\theta)}{\mathrm{d}r_0}\,\mathbf{e}_\theta + \frac{\mathrm{d}U_{xd}}{\mathrm{d}r_0}u\,\mathbf{e_{xd}}\,. \tag{10}$$

This system of equations will be referred later as:

$$\mathcal{L}(\mathbf{u}, \rho, p) = \begin{pmatrix} S_\rho \\ \mathbf{S} \end{pmatrix}. \tag{11}$$

3 Acoustic analogy in an annular duct with swirling mean flow

These developments can be seen as a generalisation of Ffowcs Williams & Hawkings' acoustic analogy [5] to swirling mean flow medium with duct walls, or as a generalisation of Goldstein formulation with uniform mean flow in a circular duct [4] to an annular duct with swirling mean flow.

3.1 Sources terms

Let Σ be the set of the B blade surfaces $\Sigma_B = \bigcup_{j=0:B-1}\Sigma_{B,j}$ and of the duct surfaces $\Sigma_D = \Sigma_{Hub}\bigcup\Sigma_{Tip}$: $\Sigma = \Sigma_B\bigcup\Sigma_D$, and $\mathbf{v}^\Sigma$ be the surface speed. The surface Σ can be defined by:

$$\begin{cases} f(\mathbf{x_d}, t) = 0 & \text{on } \Sigma(t) \\ f(\mathbf{x_d}, t) > 0 & \text{in the fluid, volume } \mathcal{V}(t) \\ f(\mathbf{x_d}, t) < 0 & \text{within } \Sigma(t)\,, \end{cases} \tag{12}$$

with $\boldsymbol{\nabla} f = \mathbf{n}$. A sketch of the problem with the notations is plotted in Figure 1. For any variable φ defined in the fluid, it is possible to define the generalised function $\widetilde{\varphi}$ defined in the whole space V and equal to φ inside the fluid ($\mathcal{V}$), and zero outside. It can be obtained by multiplying φ by $H(f)$. By definition of f the following relations apply (see for instance Jones [6] or Farassat [7]):

$$\frac{\partial f(\mathbf{x_d}, t)}{\partial t} + \mathbf{v}^\Sigma\cdot\mathbf{n} = 0 \quad , \quad \boldsymbol{\nabla} H(f(\mathbf{x_d}, t)) = \delta(f)\mathbf{n}\,, \tag{13}$$

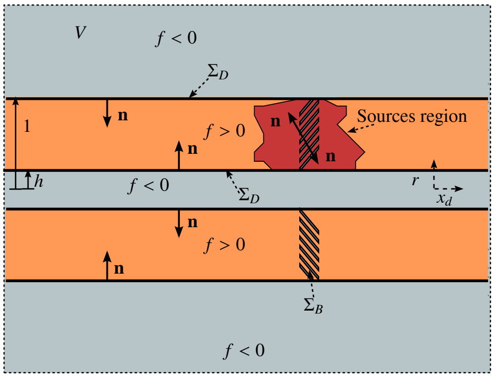

Figure 1: Sketch of the problem and the notations

where H is the Heaviside function. Then, multiplying Eqs. (2), (3), (4) and (5) by $H(f)$ and using the above relations allows us to write the problem in the following generalised form:

$$\mathcal{L}\left(\widetilde{\mathbf{u}}, \widetilde{\rho}, \widetilde{p}\right) = \begin{pmatrix} \widetilde{S}_\rho \\ \widetilde{\mathbf{S}} \end{pmatrix} + \begin{pmatrix} S_{FWH,\rho} \\ \mathbf{S_{FWH}} \end{pmatrix} \delta(f), \qquad (14)$$

where the operator $\mathcal{L}$ is applied to the generalised variables, $\widetilde{S}_\rho$ and $\widetilde{\mathbf{S}}$ stand for the generalised source terms given by replacing the physical variables in Eq. (7) and Eq. (9) by the generalised ones and where $S_{FWH,\rho}\delta(f)$ and $\mathbf{S_{FWH}}\delta(f)$ are additional surface source terms caused by the use of generalised function and the presence of the surface Σ. They are defined by:

$$\begin{aligned} \mathbf{S_{FWH}} &= \left(S_{FWH,r}, S_{FWH,\theta}, S_{FWH,x}\right) = \overline{\overline{\mathbf{L}}} \cdot \mathbf{n} \\ &= \rho_{to}\mathbf{u}\left(\mathbf{u} + \mathbf{U} - \mathbf{V}^\Sigma\right) \cdot \mathbf{n} + p\mathbf{n} - \overline{\overline{\tau}} \cdot \mathbf{n}, \end{aligned} \qquad (15)$$

and

$$S_{FWH,\rho} = \rho_{to}\left(\mathbf{u} + \mathbf{U} - \mathbf{V}^\Sigma\right) \cdot \mathbf{n} - \rho_0\left(\mathbf{U} - \mathbf{V}^\Sigma\right) \cdot \mathbf{n} = \mathbf{Q} \cdot \mathbf{n}. \qquad (16)$$

These source terms can be seen as the generalisation of what is obtained in a medium at rest by Ffowcs Williams & Hawkings [5], or in uniformly moving medium, e.g. by Najafy-Yazdi *et al.* [8], to a more general medium, possibly sheared or swirling.

3.2 Equation for the pressure fluctuation p

First, the tangential v and axial w fluctuating velocity components are eliminated from the system by applying the convective derivative operator $\mathrm{D}_0/\mathrm{D}t$ to Eq. (2) and Eq. (4) and removing v and w by using the tangential derivative $\partial/(r\partial\theta)$ of Eq. (3) and the axial derivative $\partial/\partial x_d$ of Eq. (5). Eq. (2) then reads:

$$\frac{\mathrm{D}_0}{\mathrm{D}t}\left[\frac{\partial u}{\partial r} + \left(\frac{1}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}r} + \frac{1}{r}\right)u\right] - \frac{\mathrm{d}(rU_\theta)}{r\mathrm{d}r}\frac{\partial u}{r\partial\theta} - \frac{\mathrm{d}U_{xd}}{\mathrm{d}r}\frac{\partial u}{\partial x_d} = \frac{1}{\rho_0}\left[\frac{\partial^2 p}{r^2\partial\theta^2} + \frac{\partial^2 p}{\partial x_d^2} - \frac{1}{c_0^2}\frac{\mathrm{D}_0^2 p}{\mathrm{D}t^2}\right], \qquad (17)$$

and Eq. (4):

$$\mathcal{M}(p) = \rho_0\,\mathcal{D}(u), \qquad (18)$$

with

$$\mathcal{M}(p) = \frac{\mathrm{D}_0}{\mathrm{D}t}\left(\frac{\partial p}{\partial r}\right) + 2\frac{U_\theta}{r^2}\frac{\partial p}{\partial\theta} - \frac{U_\theta^2}{r c_0^2}\frac{\mathrm{D}_0 p}{\mathrm{D}t},$$
$$\text{and } \mathcal{D}(p) = -\frac{\mathrm{D}_0^2 p}{\mathrm{D}t^2} - 2\frac{U_\theta}{r^2}\frac{\mathrm{d}(rU_\theta)}{\mathrm{d}r}p. \qquad (19)$$

Applying the operator $\mathcal{D}$ to Eq. (18), and using the relation

$$\frac{\partial}{\partial r}\left(\frac{\mathrm{D}_0}{\mathrm{D}t}\right) = \frac{\mathrm{D}_0}{\mathrm{D}t}\left(\frac{\partial}{\partial r}\right) + \frac{\mathrm{d}U_{xd}}{\mathrm{d}r}\frac{\partial}{\partial x_d} + \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{U_\theta}{r}\right)\frac{\partial}{\partial\theta}, \qquad (20)$$

on the one hand, taking the derivation of Eq. (19) with respect to r, then applying the convective derivative operator to the result, and finally the operator $\mathcal{D}$, on the other hand, it is possible to show that the pressure fluctuation p satisfies:

$$\mathcal{F}(\widetilde{p}) = \widetilde{\mathbb{S}} + \mathbb{S}_{FWH}, \qquad (21)$$

where $\mathcal{F}$ is the sixth order operator in space and time:

$$\begin{aligned} \mathcal{F}(\widetilde{p}) = {} & \mathcal{D}\left[\frac{\mathrm{D}_0}{\mathrm{D}t}\left(\frac{\partial\left[\mathcal{M}(\widetilde{p})\right]}{\partial r}\right)\right] \\ & - \left[\frac{\partial^2}{r^2\partial\theta^2} + \frac{\partial^2}{\partial x_d^2} - \frac{1}{c_0^2}\frac{\mathrm{D}_0^2}{\mathrm{D}t^2}\right]\mathcal{D}^2(\widetilde{p}) \\ & + \left\{2\frac{\mathrm{D}_0^2}{\mathrm{D}t^2}\left[\frac{\mathrm{d}U_{xd}}{\mathrm{d}r}\frac{\partial}{\partial x_d} + \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{U_\theta}{r}\right)\frac{\partial}{\partial\theta}\right]\right. \\ & + \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{2\,U_\theta}{r^2}\frac{\mathrm{d}(r\,U_\theta)}{\mathrm{d}r}\right]\frac{\mathrm{D}_0}{\mathrm{D}t} \\ & \left. + \left[\frac{1}{r}\frac{\mathrm{D}_0}{\mathrm{D}t} - \frac{\mathrm{d}(rU_\theta)}{r\mathrm{d}r}\frac{\partial}{r\partial\theta} - \frac{\mathrm{d}U_{xd}}{\mathrm{d}r}\frac{\partial}{\partial x_d}\right]\mathcal{D}\right\}\mathcal{M}(\widetilde{p}), \end{aligned} \qquad (22)$$

$$\widetilde{\mathbb{S}} + \mathbb{S}_{FWH} = \mathcal{A}\left(\widetilde{\mathbb{S}}_1 + \mathbb{S}_{FWH,1}\right) + \mathcal{D}^2\left(\widetilde{\mathbb{S}}_2 + \mathbb{S}_{FWH,2}\right), \qquad (23)$$

with the operator:

$$\begin{aligned} \mathcal{A} = {} & \mathcal{D}\left[\frac{\mathrm{D}_0}{\mathrm{D}t}\left(\frac{1}{r} + \frac{\partial}{\partial r}\right) - \frac{\mathrm{d}(rU_\theta)}{r\mathrm{d}r}\frac{\partial}{r\partial\theta} - \frac{\mathrm{d}U_{xd}}{\mathrm{d}r}\frac{\partial}{\partial x_d}\right] \\ & + 2\frac{\mathrm{D}_0^2}{\mathrm{D}t^2}\left[\frac{\mathrm{d}U_{xd}}{\mathrm{d}r}\frac{\partial}{\partial x_d} + \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{U_\theta}{r}\right)\frac{\partial}{\partial\theta}\right] \\ & + \frac{\mathrm{d}}{\mathrm{d}r}\left[\frac{2\,U_\theta}{r^2}\frac{\mathrm{d}(r\,U_\theta)}{\mathrm{d}r}\right]\frac{\mathrm{D}_0}{\mathrm{D}t}, \end{aligned} \qquad (24)$$

and

$$\widetilde{\mathbb{S}}_1 = \frac{\mathrm{D}_0\widetilde{S}_r}{\mathrm{D}t} + 2\frac{U_\theta}{r}\widetilde{S}_\theta \quad , \quad \widetilde{\mathbb{S}}_2 = \frac{\mathrm{D}_0\widetilde{S}_\rho}{\mathrm{D}t} - \frac{1}{r}\frac{\partial\widetilde{S}_\theta}{\partial\theta} - \frac{\partial\widetilde{S}_x}{\partial x_d}, \qquad (25)$$

$$\begin{aligned} \mathbb{S}_{FWH,1} &= \frac{\mathrm{D}_0\left[S_{FWH,r}\delta(f)\right]}{\mathrm{D}t} + 2\frac{U_\theta}{r}S_{FWH,\theta}\delta(f), \\ \mathbb{S}_{FWH,2} &= \frac{\mathrm{D}_0\left[S_{FWH,\rho}\delta(f)\right]}{\mathrm{D}t} - \frac{\partial\left[S_{FWH,\theta}\delta(f)\right]}{r\,\partial\theta} \\ & \quad - \frac{\partial\left[S_{FWH,x}\delta(f)\right]}{\partial x_d}. \end{aligned} \qquad (26)$$

3.3 Green's function

Let G be the Green's function tailored to an annular duct of axis x_d with swirling mean flow, namely, solution of:

$$\mathcal{F}(G) = -\delta\left(x_d - x_{d0}\right)\frac{\delta\left(r - r_0\right)}{r_0}\sum_{n\in\mathbb{Z}}\delta\left(\theta - \theta_0 - 2\pi\,n\right)\delta\left(t - t_0\right), \qquad (27)$$

with the boundary condition that the normal velocity associated with G is zero on the duct walls. It can be shown that G is equivalently defined by:

$$G\left(\mathbf{x_d}, t|\,\mathbf{x_{d0}}, t_0\right) = \int\sum_{m\in\mathbb{Z}}\int\widehat{G}_m\left(r|\,k, \omega, \mathbf{x_{d0}}, t_0\right)\,\mathrm{e}^{\mathrm{i}kx_d + \mathrm{i}m\theta - \mathrm{i}\omega t}\,\mathrm{d}k\mathrm{d}\omega \qquad (28)$$

with

$$\widehat{G}_m\left(r|\,k, \omega, \mathbf{x_{d0}}, t_0\right) = -\frac{\widehat{p}_{G,m}\left(r|\,k, \omega, \mathbf{x_{d0}}, t_0\right)}{D_{m,k}(r_0)\,\Lambda_{m,k}(r_0)^2}, \qquad (29)$$

and $\widehat{p}_{G,m}$ the solution of:

$$D_{m,k}\,\mathcal{L}\left(\widehat{p}_{G,m}\right) = -\frac{\delta\left(r-r_0\right)}{\left(2\pi\right)^3\ r_0}\,\mathrm{e}^{-\mathrm{i}kx_0+\mathrm{i}\omega t_0-\mathrm{i}m\theta_0}\,, \tag{30}$$

$$B_m\,\widehat{p}_{G,m} + \frac{\partial \widehat{p}_{G,m}}{\partial r} = 0 \quad \text{at } r = h \text{ and } r = 1\,, \tag{31}$$

where the operator $\mathcal{L}$ is:

$$\mathcal{L}\left(\widehat{p}\right) = \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r}{D_{m,k}}\left(B_m\widehat{p} + \frac{\mathrm{d}\widehat{p}}{\mathrm{d}r}\right)\right) - \frac{2\,m\,U_\theta}{\Lambda_{m,k}\,r^2\,D_{m,k}}\left(B_m\,\widehat{p} + \frac{\mathrm{d}\widehat{p}}{\mathrm{d}r}\right) + \frac{1}{\Lambda_{m,k}^2}\left(\frac{\Lambda_{m,k}^2}{c_0^2} - \frac{m^2}{r^2} - k^2\right)\widehat{p}\,, \tag{32}$$

and

$$B_m(r) = \frac{2\,m\,U_\theta}{\Lambda_{m,k}\,r^2} - \frac{U_\theta^2}{r\,c_0^2}\,. \tag{33}$$

When there is no swirl ($U_\theta = 0$), $B_m = 0$ and the zero normal velocity condition on the duct walls $u_r = 0$ reduces to the well known condition: $\partial p/\partial r = 0$. Using the general theory of differential equations (e.g. see Bender & Orszag [9]) to solve the Green's function, it is possible to show that:

$$\widehat{p}_{G,m}\left(k, r, \omega | x_{d0}, r_0, \theta_0, t_0\right) = -\frac{\mathrm{e}^{-\mathrm{i}kx_0+\mathrm{i}\omega t_0-\mathrm{i}m\theta_0}}{\left(2\pi\right)^3\ r_0\ K(k, r_0)}\begin{cases}\widehat{p}_{G,m,2}\left(k, r_0\right)\widehat{p}_{G,m,1}\left(k, r\right)\,, & r \leqslant r_0\,,\\ \widehat{p}_{G,m,1}\left(k, r_0\right)\widehat{p}_{G,m,2}\left(k, r\right)\,, & r > r_0\,,\end{cases} \tag{34}$$

where $\widehat{p}_{G,m,1}$ and $\widehat{p}_{G,m,2}$ are two solutions of the homogeneous equation $\mathcal{L}\left(\widehat{p}\right) = 0$ satisfying respectively the boundary conditions:

$$\begin{cases} B_m\left(h\right)\widehat{p}_{G,m,1}\left(h\right) + \dfrac{\partial \widehat{p}_{G,m,1}}{\partial r}\left(h\right) &= 0\,,\\ h\widehat{p}_{G,m,1}\left(h\right) &= 1\,,\end{cases} \tag{35}$$

and

$$\begin{cases} \widehat{p}_{G,m,2}\left(1\right) &= 1\,,\\ B_m\left(1\right)\widehat{p}_{G,m,2}\left(1\right) + \dfrac{\partial \widehat{p}_{G,m,2}}{\partial r}\left(1\right) &= 0\,,\end{cases} \tag{36}$$

and where

$$K\left(k, r_0\right) = \frac{\left(\rho_0 D\right)\left(r_0\right)}{r_0\left(\rho_0 D\right)\left(h\right)}\left[\frac{\partial \widehat{p}_{G,m,2}}{\partial r}(k, h) + B_m(k, h)\widehat{p}_{G,m,2}(k, h)\right]. \tag{37}$$

For each studied frequency ω and each azimuthal mode order m, the eigenvalue problem is first solved using a pseudo-spectral method using both the Chebyshev collocation grid and the Chebyshev staggered grid as proposed and detailed by Khorrami [10] to yield the axial wave-numbers $k_{m,n}^{\pm}$ of the sonic and nearly-convected modes. The critical layer is also investigated. An example of eigenvalues and critical-layer is plotted in Figure 2. The integration over the wavenumber k to yield the pressure field in the space domain is split in two parts. The contribution of each sonic mode is computed using the residue theorem, whereas the contribution of the nearly-convected modes and of the critical layer is obtained from a numerical contour in the $\mathbb{C} - k$–plane surrounding them as in Figure 2. In both cases, the functions $\widehat{p}_{G,m,i}$ are found as solutions of an initial value problem for a system of two first order differential equations using for instance the fortran routine `DVODE_F90.f`.

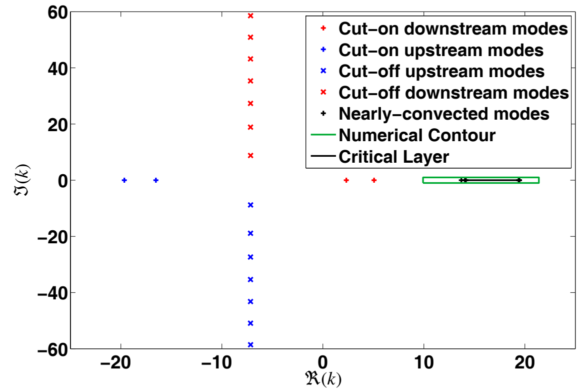

Figure 2: Sonic and nearly-convected modes, critical layer and numerical integration contour in the $\mathbb{C} - k$–plane in the case of §3.1 of Heaton [3].

3.4 Pressure field given by the acoustic analogy

Let us define a generalised Green's function $\widetilde{G}$ equal to G in the region defined by $f \geqslant 0$, and zero in region defined by $f < 0$. Then, p is given by:

$$p(x_d, r, \theta, t) = \int\!\!\!\iiint_V \widetilde{G}\left(\mathbf{x_d}, t | \mathbf{x_{d0}}, t_0\right)\widetilde{\mathbb{S}}\left(\mathbf{x_{d0}}, t_0\right)\mathrm{d}V_0\mathrm{d}t_0 + \int\!\!\!\iiint_V \widetilde{G}\left(\mathbf{x_d}, t | \mathbf{x_{d0}}, t_0\right)\left(\mathcal{A}_0(\mathbb{S}_{FWH,1}) + \mathcal{D}_0^2\left(\mathbb{S}_{FWH,2}\right)\right)\mathrm{d}V_0\mathrm{d}t_0 \tag{38}$$

where the sources terms $\widetilde{\mathbb{S}}$, $\mathbb{S}_{FWH,1}$ and $\mathbb{S}_{FWH,2}$ are evaluated at the source time t_0 and space coordinates $\mathbf{x_{d0}}$, and the subscript 0 on the operators $\mathcal{A}$ and $\mathcal{D}$ stand for derivatives with respect to the source time t_0 and space coordinates $\mathbf{x_{d0}}$. This relation is true even if the Green's function is zero out of the duct because the volume sources are zero in this region too. The sources FWH can be isolated from any derivative by successive integrations by parts. Given that the integration is performed over the whole space V and not the fluid domain $\mathcal{V}$, the integration by parts does not lead to surface terms contributions. Once the $S_{FWH,x}\delta(f)$,... terms have been isolated it is possible to reduce the integration to the surface Σ to give after rearrangement:

$$\begin{aligned}p(x_d, r, \theta, t) = &\int\iiint_V \widetilde{G}\left(\mathbf{x_d}, t | \mathbf{x_{d0}}, t_0\right)\widetilde{\mathbb{S}}\left(\mathbf{x_{d0}}, t_0\right)\mathrm{d}V_0\mathrm{d}t_0\\ &+ \iiint_{\Sigma(t_0)} \mathbf{S_{FWH}}\cdot\boldsymbol{\nabla}\left(\mathcal{D}_0^2\left(G\right)\right) - S_{FWH,\rho}\frac{\mathrm{D}_0\left(\mathcal{D}_0^2\left(G\right)\right)}{\mathrm{D}t_0}\mathrm{d}\Sigma_0(t_0)\mathrm{d}t_0\\ &+ \iiint_{\Sigma(t_0)} 2\frac{U_\theta}{r_0}\mathcal{R}_{0,1}\left(G\right)\times S_{FWH,\theta}\mathrm{d}\Sigma_0(t_0)\mathrm{d}t_0\\ &+ \iiint_{\Sigma(t_0)} S_{FWH,r}\times\mathcal{R}_{0,2}\left(G\right)\mathrm{d}\Sigma_0(t_0)\mathrm{d}t_0\,,\end{aligned} \tag{39}$$

with

$$\begin{aligned}\mathcal{R}_{0,1}\left(\widetilde{G}\right) = &\frac{\partial}{\partial r_0}\left(\frac{\mathrm{D}_0}{\mathrm{D}t_0}\left[\mathcal{D}_0\left(\widetilde{G}\right)\right]\right) - \frac{\mathrm{d}}{\mathrm{d}r_0}\left[\frac{2\,U_\theta}{r_0^2}\frac{\mathrm{d}(r_0\,U_\theta)}{\mathrm{d}r_0}\right]\frac{\mathrm{D}_0\widetilde{G}}{\mathrm{D}t_0}\\ &+ \left[\frac{\mathrm{d}(r_0 U_\theta)}{r_0\mathrm{d}r_0}\frac{\partial}{r_0\,\partial\theta_0} + \frac{\mathrm{d}U_{xd}}{\mathrm{d}r_0}\frac{\partial}{\partial x_{d0}}\right]\mathcal{D}_0\left(\widetilde{G}\right)\\ &-2\left[\frac{\mathrm{d}U_{xd}}{\mathrm{d}r_0}\frac{\partial}{\partial x_{d0}} + \frac{\mathrm{d}}{\mathrm{d}r_0}\left(\frac{U_\theta}{r_0}\right)\frac{\partial}{\partial\theta_0}\right]\frac{\mathrm{D}_0^2\widetilde{G}}{\mathrm{D}t_0^2}\,,\end{aligned} \tag{40}$$

and

$$\mathcal{R}_{0,2}\left(G\right) = U_\theta\mathcal{R}_{0,3}\left(G\right) + \frac{\mathrm{d}U_\theta}{\mathrm{d}r_0}\mathcal{R}_{0,4}\left(G\right) + \frac{\mathrm{d}U_{xd}}{\mathrm{d}r_0}\mathcal{R}_{0,5}\left(G\right)\,, \tag{41}$$

with

$$\mathcal{R}_{0,3}(G) = -\frac{8}{r_0^2}\frac{\mathrm{d}(r_0U_\theta)}{\mathrm{d}r_0}\frac{\mathrm{d}}{\mathrm{d}r_0}\left[\frac{U_\theta}{r_0^2}\frac{\mathrm{d}(r_0U_\theta)}{\mathrm{d}r_0}\right]G$$
$$-\frac{2}{r_0^2}\frac{\mathrm{d}(r_0U_\theta)}{\mathrm{d}r_0}\frac{\partial}{\partial r_0}\left[\frac{\mathrm{D}_0^2G}{\mathrm{D}t_0^2}\right] - 4U_\theta\left[\frac{1}{r_0^2}\frac{\mathrm{d}(r_0U_\theta)}{\mathrm{d}r_0}\right]^2\frac{\partial G}{\partial r_0}$$
$$+4\frac{U_\theta}{r_0^4}\frac{\mathrm{d}(r_0U_\theta)}{\mathrm{d}r_0}\frac{\partial}{\partial\theta_0}\left[\frac{\mathrm{D}_0G}{\mathrm{D}t_0}\right], \quad (42)$$

$$\mathcal{R}_{0,4}(G) = \frac{2}{r_0}\frac{\partial}{\partial\theta_0}\left(\frac{\mathrm{D}_0^3G}{\mathrm{D}t_0^3}\right), \quad (43)$$

and

$$\mathcal{R}_{0,5}(G) = 2\frac{\partial}{\partial x_{d0}}\left(\frac{\mathrm{D}_0^3G}{\mathrm{D}t_0^3}\right). \quad (44)$$

The volume term can also be reduced to an integration over the fluid domain $\mathcal{V}$. This is not detailed here for conciseness. The result will only be discussed in particular cases. The first two surface integrals in Eq. (39) are the analog to what is obtained in uniform mean flow but with $\mathcal{D}_0^2(G)$ instead of a Green's function in uniform mean flow G_{unif}. The third term is only non-zero because of the swirl. The fourth terms is the sum of three terms and is non-zero if at least one of the three following flow parameters $\mathrm{d}U_\theta/\mathrm{d}r_0$, $\mathrm{d}U_{xd}/\mathrm{d}r_0$ or U_θ is non-zero.

The surface terms over the duct surface Σ_D is first considered. The duct walls are stationnary, and are supposed to be impermeable which imposes: $\mathbf{Q}\cdot\mathbf{n} = 0$ and $\overline{\overline{\boldsymbol{L}}}\cdot\mathbf{n} = p\mathbf{n} - \overline{\overline{\boldsymbol{\tau}}}\cdot\mathbf{n}$. If in addition the stress tensor effects are neglected $\overline{\overline{\boldsymbol{L}}}\cdot\mathbf{n} = p\mathbf{n}$ and $\mathbf{S_{FWH}} = p\mathbf{n}$ on Σ_D. Finally, if the duct is supposed to be straight: $(n_r, n_\theta, n_x) = (1, 0, 0)$, and the duct surface term reduces to:

$$\int\!\!\int\!\!\int_{\Sigma_D} p\left[\frac{\partial}{\partial r_0}\left[\mathcal{D}_0^2(G)\right] + \mathcal{R}_{0,2}(G)\right]\mathrm{d}\Sigma_0(t_0)\mathrm{d}t_0. \quad (45)$$

It is possible to show that the time and axial coordinate Fourier transform and the Fourier series in θ of the integrand is exactly zero on the duct surface if G is the tailored Green's function to the annular duct. Be careful however, that this result is not obvious since G is defined with boundary conditions at $r = h$ and $r = 1$ involving $\partial G/\partial r$ whereas the boundary condition to ensure the cancellation of the integrand of the duct surface terms (Eq. (45)) requires boundary conditions at $r_0 = h$ and $r_0 = 1$ involving $\partial G/\partial r_0$. Finally, the use of the tailored Green's function indeed allows to remove any surface contribution of the duct walls, Σ in Eq. (39) then reduces to the blades surfaces Σ_B.

4 Particular configurations

4.1 Sheared flow with no surfaces: Lilley's formulation

If there is no swirl ($U_\theta = 0$), the operator $\mathcal{F}$ reduces to:

$$\mathcal{F}(p) = \frac{\mathrm{D}_0^3}{\mathrm{D}t_0^3}\left[\frac{\mathrm{D}_0}{\mathrm{D}t_0}\left(\frac{1}{c_0^2}\frac{\mathrm{D}_0^2p}{\mathrm{D}t_0^2} - \nabla^2p\right) + 2\frac{\mathrm{d}U_{xd}}{\mathrm{d}r_0}\frac{\partial^2p}{\partial x_{d0}\,\partial r_0}\right]. \quad (46)$$

The volume source terms $\mathbb{S}$ become:

$$\mathbb{S} = \frac{\mathrm{D}_0^3}{\mathrm{D}t_0^3}\left\{\frac{\mathrm{D}_0}{\mathrm{D}t_0}\left(\frac{\mathrm{D}_0S_\rho}{\mathrm{D}t} - \boldsymbol{\nabla}\cdot\mathbf{S}\right) + 2\frac{\mathrm{d}U_{xd}}{\mathrm{d}r_0}\frac{\partial S_r}{\partial x_{d0}}\right\}. \quad (47)$$

Let's define the Lighthill like tensor

$$T_{ij} = \rho_{to}\,u_i\,u_j + (p - c_0^2\rho)\delta_{ij} - \tau_{ij} = \rho_{to}\,u_i\,u_j - \tau_{ij} + c_0^2Z\delta_{ij}. \quad (48)$$

Then, if the heat conduction and viscous dissipation effects are neglected: $\overline{\overline{\boldsymbol{\tau}}} = 0$, $Z = (p - c_0^2\rho)/c_0^2 = 0$, and the pressure field is solution of:

$$\frac{\mathrm{D}_0^3}{\mathrm{D}t_0^3}\left\{\frac{\mathrm{D}_0}{\mathrm{D}t_0}\left(\frac{1}{c_0^2}\frac{\mathrm{D}_0^2p}{\mathrm{D}t_0^2} - \nabla^2p\right) + 2\frac{\mathrm{d}U_1}{\mathrm{d}y_2}\frac{\partial^2p}{\partial y_1\,\partial y_2}\right\} =$$
$$\frac{\mathrm{D}_0^3}{\mathrm{D}t_0^3}\left\{\frac{\mathrm{D}_0}{\mathrm{D}t_0}\left[\frac{\partial^2T_{ij}}{\partial y_i\,\partial y_j}\right] - 2\frac{\mathrm{d}U_1}{\mathrm{d}y_2}\frac{\partial^2(\rho_{to}u_2u_i)}{\partial y_1\,\partial y_i}\right\}, \quad (49)$$

in Cartesian coordinates (1: streamwise, 2: normal to the shear layer). This is exactly the third order convective derivative $\mathrm{D}_0^3/\mathrm{D}t_0^3$ of Lilley's equation assuming that the viscous and heat conductions effects are negligible and the fluctuating pressure is sufficiently small so that the operator $\Pi = \log(p/p_0)$ of Lilley can be approximate by $\Pi \approx (p/p_0)$ (e.g. section 6 of [4], [11]).

4.2 Uniform mean flow in an annular duct: Goldstein's formulation

With uniform mean flow, Eq. (39) reduces to

$$p(x_d, r, \theta, t) = \int\!\!\int\!\!\int\!\!\int_{\mathcal{V}(t_0)} Z\left\{\frac{\mathrm{D}_0^2}{\mathrm{D}t_0^2} - c_0^2\boldsymbol{\nabla}^2\right\}\left[\frac{\mathrm{D}_0^4}{\mathrm{D}t_0^4}(G)\right]\mathrm{d}V_0\mathrm{d}t_0$$
$$+\int\!\!\int\!\!\int\!\!\int_{\mathcal{V}(t_0)} T_{ij}\frac{\partial^2}{\partial y_i\,\partial y_j}\left[\frac{\mathrm{D}_0^4}{\mathrm{D}t_0^4}(G)\right]\mathrm{d}V_0\mathrm{d}t_0$$
$$+\int\!\!\int\!\!\int_{\Sigma_B(t_0)} \overline{\overline{\boldsymbol{L}}}\cdot\mathbf{n}\cdot\boldsymbol{\nabla}\left(\frac{\mathrm{D}_0^4G}{\mathrm{D}t_0^4}\right) - \mathbf{Q}\cdot\mathbf{n}\frac{\mathrm{D}_0}{\mathrm{D}t_0}\left(\frac{\mathrm{D}_0^4G}{\mathrm{D}t_0^4}\right)\mathrm{d}\Sigma_0(t_0)\mathrm{d}t_0. \quad (50)$$

The first term is caused by the fact that the pressure is evaluated instead of the density as is usually done (e.g. [4, 5]). Besides, in this uniform mean flow case the operator $\mathcal{F}$ corresponds to apply the fourth order particular derivative $-\mathrm{D}_0^4/\mathrm{D}t_0^4$ to the wave operator in uniform mean flow. Then G is solution of:

$$-\frac{\mathrm{D}_0^4}{\mathrm{D}t^4}\left[\boldsymbol{\nabla}^2 - \frac{1}{c_0^2}\frac{\mathrm{D}_0^2}{\mathrm{D}t^2}\right]G(\mathbf{x}_\mathrm{d}, t|\mathbf{x}_{\mathrm{d}0}, t_0) = -\delta(\mathbf{x}_d - \mathbf{x}_{d0})\,\delta(t - t_0), \quad (51)$$

with $\partial G/\partial r = 0$ in $r = h$ and $r = 1$. Given that in this particular case, the convective derivative and the radial derivative commutes, $Y = -\mathrm{D}_0^4G/\mathrm{D}t_0^4$ is solution of:

$$\left[\boldsymbol{\nabla}^2 - \frac{1}{c_0^2}\frac{\mathrm{D}_0^2}{\mathrm{D}t^2}\right]Y(\mathbf{x}_\mathrm{d}, t|\mathbf{x}_{\mathrm{d}0}, t_0) = -\delta(\mathbf{x}_d - \mathbf{x}_{d0})\,\delta(t - t_0), \quad (52)$$

with $\partial Y/\partial r = 0$ in $r = h$ and $r = 1$, namely Y is exactly the Green's function used in uniform mean flow, and the pressure field expression Eq. (50) is the analog in an annular duct (used for instance by [12, 13]) to that given by Goldstein in the circular duct case for the density field.

5 Application to ducted fan noise

The acoustic analogy detailed in section 3 can be used to deal with several noise-generation mechanisms and the associated propagation in a rotor/stator fan stage. For instance, it allows us to account for the downstream swirling mean-flow effect on the rotor trailing-edge noise or for the upstream swirling mean-flow effect on the rotor-stator interaction noise. If the volume sources are neglected as is usually done in these contexts in subsonic regimes, and if the blades are supposed to be radial (no lean angle, $n_r = 0$), then p (Eq. (39)) is:

$$p(x_d, r, \theta, t) = \int\!\!\int\!\!\int_{\bigcup_j \Sigma_{B,j}(t_0)} p(\mathbf{x_{d0}}, t_0)\mathcal{T}_0(G(\mathbf{x_d}, t|\mathbf{x_{d0}}, t_0))\mathrm{d}\Sigma_0(t_0)\mathrm{d}t_0 \,, \tag{53}$$

with

$$\mathcal{T}_0(G) = \left[n_{x,j}\mathcal{D}_0^2 \frac{\partial G}{\partial x_{d0}} + n_{\theta,j}\left(\frac{\mathcal{D}_0^2}{r_0}\frac{\partial G}{\partial \theta_0} + 2\frac{U_\theta}{r_0}\mathcal{R}_{0,1}(G) \right) \right] . \tag{54}$$

It is then possible to use the same strategy as was done up to now (e.g. [4, 12, 13]) to compute the noise generated, by simply providing the pressure jump (assuming flat plate) or the blade pressure distribution by an analytical model or a CFD simulation to compute the radiated noise.

6 Conclusion

An acoustic analogy in a swirling medium has been derived, extending Ffowcs Williams & Hawkings and Goldstein's acoustic analogies. It has been shown to reduce to Lilley's equation in the particular case of a sheared flow without surface, and to Goldstein's acoustic analogy in the particular case of a uniform mean flow in a duct. The Green's function tailored to the annular duct with swirling mean flow has been derived. In the particular context of fan noise, the pressure field has been written as the sum over the surfaces of all the blades of the pressure distribution multiplied by an operator acting on the Green's function.

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References

[1] V. V. Golubev, H. M. Atassi, “Unsteady swirling flows in annular cascades, Part 2 : Aerodynamic blade response incident disturbance”, *AIAA J.*, **38**(7), 1150-1158 (2000)

[2] H. M. Atassi, A. A. Ali, O. V. Atassi, I. V. Vinogradov, “Scattering of incidence disturbances by an annular cascade in a swirling flow”, *J. Fluid Mech.*, **499**, 111-138 (2004)

[3] C. J. Heaton, N. Peake, “Algebraic and exponential instability of inviscid swirling flow”, *J. Fluid Mech.*, **565**, 279-318 (2006)

[4] M. E. Goldstein, *Aeroacoustics*, McGraw-Hill, New York, (1976)

[5] J. E. Ffowcs Williams, D. L. Hawkings, ”Sound generation by turbulence and surfaces in arbitrary motion”, *Phil. Trans. R. Soc. Lond. A*, **264**, 321-342 (1969)

[6] D. S. Jones, *The theory of generalised functions*, Cambridge University Press, 2nd Ed. (1982)

[7] F. Farassat, ”Discontinuities in aerodynamics and aeroacoustics: The concept and applications of generalized derivatives”, *J Sound Vib.*, **55**(2), 165-193 (1977)

[8] A. Najafi-Yazdi, G. A. Brès, L. Mongeau, ”An acoustic analogy formulation for moving sources in uniformly moving media”, *Phil. Trans. R. Soc. Lond. A*, **467**, 144-165 (2011)

[9] C. M. Bender, S. A. Orszag, *Advanced Mathematical Methods for Scientists and Engineers. Asymptotic Methods and perturbation theory*, Mc Grall-Hill (1978)

[10] M. R. Khorrami, “A {C}hebyshev spectral collocation method using a staggered grid for the stability of cylindrical flows”, *Int. J. Num. Meth. Fluids*, **12**, 825-833 (1991)

[11] T. Colonius, S. K. Lele, P. Moin, “Sound generation in a mixing layer”, *J. Fluid Mech.*, **330**, 375-409, (1997)

[12] C. S. Ventres, M. A. Theobald, W. D. Mark, “Turbofan Noise Generation, Volume 1: Analysis”, *NASA Contractor Report*, **CR-167952**

[13] H. Posson, S. Moreau, “Broadband noise prediction of fan outlet guide vane using a cascade response function”, *J Sound Vib.*, **330**, 6153-6183 (2011)