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Effective wave numbers for media sustaining the propagation of three types of bulk waves and hosting a random configuration of scatterers

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Wave propagation through an isotropic host medium containing a large number of randomly and uniformly located scatterers is considered at low frequency and for low concentrations of spheres, and the dispersion relation of the coherent waves is obtained. The same problem had been addressed by Lloyd and Berry for spheres in an ideal fluid, and more recently by Linton and Martin for cylinders in an ideal fluid, and by Conoir and Norris for cylinders in an elastic solid. Here, the dispersion relation is derived in the case of spheres, and extended to that of cylinders, from the comparison of the 3d and 2d cases in an elastic solid. The host medium considered may support the propagation of P different types of bulk waves, as for example a thermo-visco-elastic medium or a poro-elastic medium ($P=3$). As in the previous works mentioned above, the hole correction of Fikioris and Waterman is taken into account, along with the quasi-crystalline approximation. The method follows exactly that used by Conoir and Norris.

1 Introduction

This paper summarizes some of the results obtained in Refs.[1,2] on the propagation of coherent waves in homogeneous media that contain distributions of either spherical ($d=3$) or cylindrical ($d=2$) inhomogeneities acting as scatterers. It is focused on the low frequency and low concentration approximations of the dispersion equation of the coherent wave associated with the “fastest wave” in the host medium (more precisely, the wave which has the smallest modulus of all complex wave numbers). In a poro-visco-elastic medium obeying Biot’s theory, for example, that would be the fast longitudinal wave.

The method used to obtain the equations that govern the coherent fields follows that of Fikioris and Waterman’s paper [2]. A harmonic plane wave propagating in the host medium is supposed to be normally incident upon some semi-infinite region $x^{(d)} > 0$ hosting a uniform concentration of scatterers. In case of spherical scatterers, $x^{(3)}$ will be equal to z , while in case of infinitely long cylinders, $x^{(2)}$ will be equal to x , and the axis of the cylinders will be parallel to the z axis, so that the problem will be of dimension $d=2$. This harmonic plane wave gives rise to a multiple scattering process and to scattered waves of different polarization types denoted by a natural number p . When averaged over all possible locations of the scatterers, the total field of a given type p represents the coherent wave of type p . For a low enough concentration of scatterers, the coherent waves are supposed to be plane waves that propagate in the same direction $x^{(d)}$ as the original incident plane wave.

In the following, we suppose that the host medium supports the propagation of P different types of waves ($P=1$ in an ideal fluid, $P=2$ in an elastic solid, $P=3$ in a thermo-visco-elastic medium or a poroelastic solid); they are numbered from $p=1$ to $p=P$, and their complex wave numbers at the given angular frequency ω are k_p , with

$$\forall p \neq 1, |k_1| < |k_p|, \quad (1)$$

so that the coherent we shall focus on in the last section corresponds to $p=1$. We shall also suppose, with no loss of generality but in order to simplify the $d=3$ dimensional problem, that the original incident plane wave, $\varphi_{inc}^{(p)}$, is also of type $p=1$, i.e. a longitudinal wave. An $e^{-i\omega t}$ time dependence of all fields, while supposed, will be omitted everywhere for the sake of brevity :

$$\varphi_{inc}^{(p)} = \delta_{p1} e^{ik_p x^{(d)}}, k_p = k_p' + ik_p'', k_p' > 0, k_p'' \geq 0. \quad (2)$$

2 Coherent fields in the framework of the Fikioris and Waterman theory

The waves are described by scalar displacement potentials. In the $d=3$ dimensional problem, these are the Debye potentials [4] of the longitudinal and shear waves : the Debye potentials of transverse waves are not taken into account for symmetry reasons linked to the longitudinal nature of the incident wave of Eq.(2), as explained in Ref.[2]. In the $d=2$ dimensional case, the scalar potentials are the displacement potential of the longitudinal waves and the z -component [5] of the vector potential of the shear waves.

The total potential associated with a wave of type p is due to the incident wave, if of the same type, and to all the scattered waves of type p , so that, considering a given number N of scatterers,

$$\varphi^{(p)}(\vec{r}) = \delta_{p1} \varphi_{inc}^{(1)} + \sum_{k=1}^N \varphi_S^{(p)}(\vec{r}; \vec{r}_k). \quad (3)$$

Here, as in Ref.[1], $\varphi_S^{(p)}(\vec{r}; \vec{r}_k)$ represents the wave (of type p) that is scattered by a scatterer centered at \vec{r}_k and observed at \vec{r} . Letting $\varphi_E^{(p)}(\vec{r}; \vec{r}_k)$ denote the field (of type p) that is exciting a scatterer centered at \vec{r}_k and observed at \vec{r} , the following equation defines the linear scattering operators $T^{(qp)}(\vec{r}_k)$ of the scatterers

$$\varphi_S^{(p)}(\vec{r}; \vec{r}_k) = \sum_{q=1}^P T^{(qp)}(\vec{r}_k) \varphi_E^{(q)}(\vec{r}; \vec{r}_k). \quad (4)$$

The exciting field acting on the k -th scatterer is the sum of the incident field and the scattered waves from all the other scatterers:

$$\varphi_E^{(p)}(\vec{r}; \vec{r}_k) = \delta_{p1} \varphi_{inc}^{(1)}(\vec{r}) + \sum_{q=1}^P \sum_{\substack{j=1 \\ j \neq k}}^N T^{(qp)}(\vec{r}_j) \varphi_E^{(q)}(\vec{r}; \vec{r}_j), \quad (5)$$

and the average exciting field on the 1st scatterer (supposed fixed), within the quasi-crystalline approximation, is given by

$$\begin{aligned} \left\langle \varphi_E^{(p)}(\vec{r}; \vec{r}_1) \right\rangle &= \delta_{p1} \varphi_{inc}^{(1)}(\vec{r}) + \\ \int d\vec{r}_j n(\vec{r}_j, \vec{r}_1) \sum_{q=1}^P T^{(qp)}(\vec{r}_j) \left\langle \varphi_E^{(q)}(\vec{r}; \vec{r}_j) \right\rangle, \end{aligned} \quad (6)$$

with the integration performed over the whole region that hosts the centers of the scatterers and $n(\vec{r}_j, \vec{r}_1)$, the conditional number density of scatterers at \vec{r}_j if one is known to be at \vec{r}_1 , given by the “hole correction” [3]:

$$n(\vec{r}, \vec{r}_j) = \begin{cases} n_0 & \text{for } \|\vec{r} - \vec{r}_j\| > b \\ 0 & \text{otherwise} \end{cases} \quad \text{with } b > 2a. \quad (7)$$

The effective potentials are expressed as infinite series that respect the symmetries of both the incident wave and the scatterers (see section II.B in Ref.[2]),

$$\begin{aligned} d=3, \\ \left\langle \varphi_E^{(p)}(\vec{r}; \vec{r}_j) \right\rangle &= \sum_{n=0}^{+\infty} A_n^{(p)}(\vec{r}_j) j_n(k_p \rho_j) P_n(\cos \theta(\vec{\rho}_j)) \\ d=2, \\ \left\langle \varphi_E^{(p)}(\vec{r}; \vec{r}_j) \right\rangle &= \sum_{n=-\infty}^{+\infty} A_n^{(p)}(\vec{r}_j) J_n(K_p \rho_j) e^{in\theta(\vec{\rho}_j)} \end{aligned} \quad (8)$$

with $\vec{\rho}_j = \vec{r} - \vec{r}_j$ and $\theta(\vec{\rho}_j) = \text{Arg}(\vec{\rho}_j)$ in both cases, and we assume the coherent waves obey the Snell-Descartes laws of refraction, *i.e.* they propagate in the $x^{(d)}$ direction:

$$\begin{aligned} d=3, A_n^{(p)}(\vec{r}_j) &= i^n (2n+1) \sum_{k=1}^P A_n^{(pk)} e^{i\xi_k X_j} \\ d=2, A_n^{(p)}(\vec{r}_j) &= i^n \sum_{k=1}^P A_n^{(pk)} e^{i\xi_k X_j} \end{aligned} \quad (9)$$

Determination of the system of equations the $A_n^{(pk)}$ amplitudes obey involves the decomposition of the incident plane wave, Eq.(2), into either spherical or cylindrical functions, the writing of the wave scattered by one scatterer as a sum of waves incident on the other scatterers *via* an addition theorem [6,7], and leads (see Refs.[1,2] for details), for identical scatterers, to

$$\begin{aligned} \forall p \in \{1, 2, \dots, P\}, \forall k \in \{1, 2, \dots, P\} \\ d=3, A_n^{(pk)} &= \frac{4n_0 \pi b}{\xi_k^2 - k_p^2} \\ \sum_{q=1}^P \sum_{\nu=0}^{+\infty} \sum_{\ell=0}^{+\infty} (-1)^\ell (2\nu+1) T_\nu^{(qp)} A_\nu^{(qk)} N_\ell^{(p)}(\xi_k) G(0, \nu | 0, n | \ell), \\ d=2, A_n^{(pk)} &= \frac{2\pi n_0}{\xi_k^2 - K_p^2} \sum_{q=1}^P \sum_{m=-\infty}^{+\infty} T_m^{(qp)} A_m^{(qk)} N_{m-n}^{(p)}(\xi_k) \end{aligned} \quad (10)$$

with

$$\begin{aligned} d=3, \\ N_\ell^{(p)}(\xi_k) &= \xi_k b j_\ell'(\xi_k b) h_\ell^{(1)}(k_p b) - k_p b j_\ell(\xi_k b) h_\ell^{(1)'}(k_p b) \\ d=2, \\ N_m^{(p)}(\xi_k) &= \xi_k b J_m'(\xi_k b) H_m^{(1)}(k_p b) - K_p b J_m(\xi_k b) H_m^{(1)'}(k_p b) \end{aligned} \quad (11)$$

and the Gaunt coefficients $G(0, \nu | 0, n | \ell)$ defined from

$$P_n(\cos \theta) P_\nu(\cos \theta) = \sum_{\ell=0}^{+\infty} G(0, \nu | 0, n | \ell) P_\ell(\cos \theta), \quad (12)$$

and $T_\nu^{(qp)}$ the modal coefficient associated with the scattering of a type q incident wave $\varphi_i^{(q)}$,

$$\begin{aligned} d=3, \varphi_i^{(q)} &= \sum_{n=0}^{+\infty} i^n (2n+1) j_n(k_{qr}) P_n(\cos \theta) \\ d=2, \varphi_i^{(q)} &= \sum_{n=-\infty}^{+\infty} i^n J_n(k_q r) e^{in\theta} \end{aligned} \quad (13)$$

into a type p wave $\varphi^{(qp)}$,

$$\begin{aligned} d=3, \\ \varphi^{(qp)} &= \sum_{n=0}^{+\infty} i^n (2n+1) T_n^{(qp)} h_n^{(1)}(k_p r) P_n(\cos \theta) \\ d=2, \\ \varphi^{(qp)} &= \sum_{n=-\infty}^{+\infty} i^n T_n^{(qp)} H_n^{(1)}(k_p r) e^{in\theta} \end{aligned} \quad (14)$$

by a single scatterer centered at $x=y=z=0$.

3 Low frequency and low concentration approximation for all coherent wave dispersion equations

The low frequency approximation corresponds to small values of $|k_p b|$, whatever that of p . In the low concentration approximation, it is assumed that ξ_p and k_p are close enough for the expansion of $\xi_p^2 - k_p^2$ in terms of powers of n_0 to be accurate enough at order 2:

$$\begin{aligned} \forall p \in \{1, 2, 3\}, \\ \xi_p^2 - k_p^2 &= d_{1p}^{(d)} n_0 + \left(d_{2p}^{(d)} + \sum_{q=1}^P \sum_{q \neq p} d_{2pq}^{(d)} \right) n_0^2 + O(n_0^3). \end{aligned} \quad (15)$$

Equations.(11) are infinite linear and homogeneous systems of equations for the unknown amplitudes $A_n^{(pk)}$;

setting their determinants to zero provides the dispersion equations of the p -type coherent wave (once again, see Ref.[2] for details), which, under both assumptions of low concentration and low frequency, leads to

$$d_{1p}^{(3)} = -4i\pi k_p^3 \sum_{n=0}^{+\infty} (2n+1) T_n^{(pp)}, \quad (16)$$

$$\begin{aligned} d_{2p}^{(3)} &= -8\pi^2 k_p^6 \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \sum_{\nu=0}^{+\infty} \\ & (2m+1)(2n+1) T_n^{(pp)} T_m^{(pp)} \nu G(0, m | 0, n | \nu) \\ d_{2pq}^{(3)} &= -16\pi^2 \frac{k_p^9}{k_q (k_p^2 - k_q^2)} \sum_{n=0}^{+\infty} \sum_{m=0}^{+\infty} \sum_{\nu=0}^{+\infty} \\ & (2m+1)(2n+1) T_n^{(qp)} T_m^{(pq)} \left(\frac{k_p}{k_q} \right)^\nu G(0, m | 0, n | \nu) \end{aligned} \quad (17)$$

for the spherical case, and to

$$d_{1p}^{(2)} = -4ik_p^2 \sum_{n=-\infty}^{+\infty} T_n^{(pp)}, \quad (18)$$

$$\begin{aligned} d_{2p}^{(2)} &= -8k_p^4 \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} |m-n| T_n^{(pp)} T_m^{(pp)} \\ d_{2pq}^{(2)} &= -16 \frac{k_p^6}{k_p^2 - k_q^2} \sum_{n=-\infty}^{+\infty} \sum_{m=-\infty}^{+\infty} \left(\frac{k_p}{k_q} \right)^{|m-n|} T_n^{(qp)} T_m^{(pq)} \end{aligned} \quad (19)$$

for the cylindrical case.

In both cases, the first order terms $d_{1p}^{(3)}$ and $d_{1p}^{(2)}$, that involve no mode conversion, do not depend on the complexity of the host medium. The second order terms that exhibit as well no mode conversions had been given by Lloyd and Berry for spheres in an ideal fluid [8,9], and by Linton and Martin for cylinders [10,1]. The coupling term for cylinders, $d_{2pq}^{(2)}$, was obtained by Conoir and Norris [1]. In Ref.[2], we used the matrix formalism developed by the latter to obtain its spherical counterpart.

While Eqs.(15-19) are the most suitable for numerical computations, the dispersion equations are most often presented in terms of the far-field scattering functions of the scatterers. This is the object of next section.

4 Low frequency and low concentration approximation of the dispersion equation of the fastest coherent wave in terms of far-field scattering functions

Equations.(20) define the far-field scattering functions $f_{pq}^{(d)}(\theta)$ from the expression of scattered fields as in

Eq.(14) at a large distance r from a single scatterer centered at $x=y=z=0$:

$$\begin{aligned} d &= 3, \\ \varphi^{(pq)}(r, \theta) &\underset{(r \rightarrow +\infty)}{\simeq} -i \frac{e^{ik_q r}}{r} \frac{f_{pq}^{(3)}(\theta)}{k_q} \\ d &= 2, \\ \varphi^{(pq)}(r, \theta) &\underset{(r \rightarrow +\infty)}{\simeq} e^{-i\pi/4} \frac{e^{ik_q r}}{\sqrt{r}} \sqrt{\frac{2}{\pi}} \frac{f_{pq}^{(2)}(\theta)}{k_q^{1/2}} \end{aligned} \quad (20)$$

with, in case of a circular cylinder, $f_{pq}^{(2)}$ an even function of θ if p and q are associated both to either longitudinal or shear waves, and an odd function if only one is associated with a longitudinal (or shear) wave.

While $d_{1p}^{(d)}$, $d_{2p}^{(d)}$, and $d_{2pq}^{(2)}$ have all been written [8-10,1] in terms of the far field form functions defined by Eq.(20), we have not managed to do the same for $d_{2pq}^{(3)}$, unless $p=1$ and obeys Eq.(1) ; this is the reason why we focus now on the coherent wave that is associated with the fastest longitudinal wave $p=1$. We found indeed [2] :

$$d_{21q}^{(3)} = \frac{8\pi^2}{k_p} \int_0^\pi \frac{f_{1q}^{(2)}(\theta) f_{q1}^{(2)}(\theta)}{(k_p^2 + k_q^2 - 2k_p k_q \cos \theta)^{3/2}} \sin \theta d\theta, \quad (21)$$

while the equivalent term in the $d=2$ case was [1] :

$$d_{21q}^{(2)} = \frac{8}{\pi} \int_0^\pi \frac{f_{1q}^{(2)}(\theta) f_{q1}^{(2)}(-\theta) + f_{q1}^{(2)}(\theta) f_{1q}^{(2)}(-\theta)}{k_1^2 + k_q^2 - 2k_1 k_q \cos \theta} d\theta. \quad (22)$$

Reducing the $d=2$ case to circular cylinders whose far field functions have the parity properties afore mentioned, and defining vectors $\vec{k}_1 = k_1 \hat{X}$, \vec{k}_q such that $\vec{k}_1 \cdot \vec{k}_q = k_1 k_q \cos \theta$, and $\vec{k}_{1q} = \vec{k}_1 - \vec{k}_q$, the dispersion equation of the coherent $p=1$ wave may be recast from Eqs.(15-22) into

$$\frac{\xi_1^2}{k_1^2} = 1 + \delta_{11}^{(d)} \frac{n_0}{k_1^d} + \delta_{21}^{(d)} \frac{n_0^2}{k_1^{2d}} + \frac{n_0^2}{k_1^d} \sum_{q=2}^p \int_{\Omega^{(d)}} \frac{\delta_{21q}^{(d)}}{k_{1d}} d\Omega^{(d)}, \quad (23)$$

with $\Omega^{(d)}$ the solid angle in the d -dimensional space.

The first order terms and the uncoupling 2nd order terms are, as given in Refs.[8-10] (in Ref.[1], all integrals corresponding to $\delta_{2p}^{(2)}$ should extend from 0 to π , instead of 2π) :

$$\begin{aligned} \delta_{11}^{(3)} &= -4i\pi f_{11}^{(3)}(0) \\ \delta_{11}^{(2)} &= -4if_{11}^{(2)}(0) \end{aligned} \quad (24)$$

$$\begin{aligned}\delta_{21}^{(3)} &= 4\pi^2 \int_0^\pi \frac{1}{\sin\left(\frac{\theta}{2}\right)} \frac{d}{d\theta} \left(f_{11}^{(3)}(\theta) \right)^2 d\theta \\ &+ 4\pi^2 \left[\left(f_{11}^{(3)}(0) \right)^2 - \left(f_{11}^{(3)}(\pi) \right)^2 \right], \quad (25) \\ \delta_{21}^{(2)} &= \frac{8}{\pi} \int_0^\pi \cot\left(\frac{\theta}{2}\right) \frac{d}{d\theta} \left(f_{11}^{(2)}(\theta) \right)^2 d\theta\end{aligned}$$

and the second order coupling terms may be written, from Eqs.(21-23), in a way that exhibits the similarities of the spherical and circular cylindrical cases :

$$\begin{aligned}\delta_{21q}^{(3)} &= 4\pi \frac{f_{1q}^{(3)}(\theta)}{k_{1d}} \frac{f_{q1}^{(3)}(-\theta)}{k_{1d}} \\ \delta_{21q}^{(2)} &= 4\sqrt{\frac{2}{\pi}} \frac{f_{1q}^{(2)}(\theta)}{k_{1d}^{1/2}} \sqrt{\frac{2}{\pi}} \frac{f_{q1}^{(2)}(-\theta)}{k_{1d}^{1/2}}.\end{aligned} \quad (26)$$

Looking back at Eqs.(24,16), one can only wonder if it is possible to write Eq.(25) in a similar way, such as

$$\begin{aligned}\delta_{21}^{(3)} &= 4\pi \int_{\Omega^{(3)}} x(\theta) f_{11}^{(3)}(\theta) f_{11}^{(3)'}(\theta) d\Omega^{(3)} \\ \delta_{21}^{(2)} &= 4 \int_{\Omega^{(2)}} x(\theta) \sqrt{\frac{2}{\pi}} f_{11}^{(2)}(\theta) \sqrt{\frac{2}{\pi}} f_{11}^{(2)'}(\theta) d\Omega^{(2)}.\end{aligned} \quad (27)$$

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