Numerical Methods for Scattering Phenomena

Bodo Nolte, Christian Fiedler, Andreas Wendlandt
Federal Armed Forces Underwater Acoustics and Marine Geophysics Research Institute, D-24148 Kiel, Klausdorfer Weg 2-24, Germany, {bodonolte, cfiedler, andreaswendlandt}@bwb.org

Ingo Schäfer
Helmut Schmidt University - University of the Federal Armed Forces Hamburg, D-22048, Holstenhofweg 85, Germany, ingo.schaefer@hsu-hh.de

Different numerical approaches for the physical phenomena of scattering waves from the obstacle due to an incident excitation are presented. They are based on different integral formulations. The wave equation is used in frequency domain (Helmholtz-Equation) for the fluid. Fluid structure interaction effects are numerical treatable as well. This phenomenon is mathematically based on the Lamé-Navier-Equation in conjunction with the fluid. The corresponding integral formulation is the Kirchhoff-Equation in frequency domain for the fluid as well as the so called Somigliana-Identity for the elastic structure. To get one numerical approach for treating of such problem the Finite Element Method including so called infinite elements at artificial boundaries can be used. Moreover we use the Boundary Element Method in different approaches because the Sommerfeld radiation condition makes sure that no reflecting waves from boundaries at infinity occur. The acoustic wave that is incident on a rigid obstacle or structure is totally reflected. Other boundary conditions could be a soft one; even the structure could be elastic. So we introduce the rigid, soft or elastic scattering of sound. One of the most disadvantages of some numerical methods FEM/BEM is the fact that we obtain difficulties when handling the very high frequency range. This is because of limited computer memory. For this high frequency range the Kirchhoff-Integral equation is divided into visible and invisible parts resp. illuminated and non-illuminated parts. With some assumptions of reflecting behaviour on the structure different approaches for the higher frequency range are obtained. Multiple scattering approaches can be compared with coupling and non-coupling BEM methods.

1 Introduction

First we consider the wave equation

\[ p_{\omega \omega} - \frac{1}{c^2} p_{\omega \omega} = 0. \]  (1)

Fourier transformation leads to the elliptic Helmholtz-equation

\[ p_{\omega \omega} + k^2 p = 0, \]  (2)

where c is the speed of sound and k is the wave number \( \frac{\omega}{c} \), \( \omega \) the circular frequency. In this paper the time harmonic form

\[ p = \hat{p} e^{i\omega t} \]  (3)

is used. That causes a Fourier transformation of the form

\[ F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt. \]  (4)

Following this notation we obtain the fundamental solution

\[ g_{\omega \omega} + k^2 g = -\delta, \quad g(\vec{X}, \vec{r}_p) = \frac{e^{-i\omega r}}{4\pi r}, \]  (5)

where \( r \) is the Euklidian distance between the load point \( P \) and the field point \( X \), the latter one is located on the boundary \( \Gamma \) (fig. 1), \( \delta \) is the Dirac-distribution.

2 Scattering from an arbitrarily shaped obstacle

2.1 A BEM-approach

To derive the integral expression for the sound field for an arbitrarily shaped obstacle, which is incident by a known sound source, the inhomogeneous Helmholtz equation

\[ p_{\omega \omega} + k^2 p + q = 0 \]  (6)

has to be modified. The inhomogeneous Helmholtz operator is weighted with a test function \( g \), which later on will become the fundamental solution (5), and is integrated over the entire domain \( \Omega \) (fig. 1)

\[ \int_{\Omega} (p_{\omega \omega} + k^2 p + q) g d\Omega = 0. \]  (7)

Twice application of Gauss divergence theorem and twice application of integration by parts lead to
\[ \int_{\Omega} \left( p_{\mu} + k^2 p + q \right) gd\Omega \]
\[ = \int_{\Gamma} \left( g \frac{\partial p}{\partial n} - p \frac{\partial g}{\partial n} \right) d\Gamma + \int_{\Omega} qgd\Omega . \quad (8) \]

Using equation (7) equation (8) yields
\[ C(P)p(P) = \int_{\Gamma} \left( g \frac{\partial p}{\partial n} - p \frac{\partial g}{\partial n} \right) d\Gamma + \int_{\Omega} qgd\Omega . \quad (9) \]

with
\[ C(P) = \begin{cases} 1 & P \in \Omega \\ 0.5 & P \in \Gamma \text{ smooth} \\ 0 & P \in \Omega^0 \end{cases} \quad (10) \]

where \( \Omega^0 \) is the complimentary domain with respect to \( \Omega \). It should be mentioned, that in equation (9) \( g \) is the fundamental solution with different arguments in the domain and on the surface integral,
\[ g(\vec{r}_Q, \vec{r}_P) \bigg|_{\Omega} \quad \text{and} \quad g(\vec{X}, \vec{r}_P) \bigg|_{\Gamma}. \]

Assume that the sound sources are located only in a part of the domain \( \Omega \), in \( q \), see fig. 1, equation (9) results in
\[ C(\Omega)p(\Omega) = \int_{\Gamma} \left( g \frac{\partial p}{\partial n} - p \frac{\partial g}{\partial n} \right) d\Gamma + \int_{\Omega} qgdq . \quad (11) \]

The numerical approach of equation (11) for every load point on the boundary leads to the BEM system of equations for the scattering problem. By using the Euler-equation
\[ \frac{\partial p}{\partial n} = -i\omega \rho v_n \quad (12) \]

one can substituted the normal particle velocity for the normal pressure derivative in equation (11), so that equation (11) can be written as
\[ \frac{1}{2} p(P) = \int_{\Gamma} \left( gi\omega \rho v_n + p \frac{\partial g}{\partial n} \right) d\Gamma + \int_{\Omega} qgdQ . \quad (13) \]

on the boundary \( \Gamma \).

The latter term in equation (13) is the incident wave. The corresponding system of equations is
\[ \mathbf{F}p = \mathbf{J}v + \mathbf{p}_{\text{incident}} . \quad (14) \]

The factor 0.5 is now located on the main diagonal terms of matrix \( \mathbf{F} \).

![Figure 1: Source \( q \) and Scatterer in the domain \( \Omega \)](image)

To obtain the integral equation for scattering phenomena one can also take a look at the Helmholtz integral equation just for the scattering sound field for an exterior problem \( p^s \) which results in
\[ C(\Omega)p^s(P) = -\int_{\Gamma} (i\omega \rho v_n^s + p^s \frac{\partial g}{\partial n}) d\Gamma . \quad (15) \]

Now let us imagine that the obstacle is not existent. The interior integral formulation just for the incident wave obtained from the sound source in the absence of the obstacle can be written as
\[ C^0(\Omega)p^i(P) = \int_{\Gamma} (i\omega \rho v_n^i + p^i \frac{\partial g}{\partial n}) d\Gamma . \quad (16) \]

where \( C^0 \) is the complimentary surface integral to \( C \) and the value of this integral is 0.5 on a smooth boundary as well. Note that in equation (16) the normal vector is still the same like in equation (15).

Subtraction of equation (16) from equation (15) and by making use of
\[ p = p^i + p^s , \quad v_n = v_n^i + v_n^s \quad \text{and} \quad C + C^0 = 1 \quad (17) \]

leads to
\[ C(\Omega)p(P) = \int_{\Gamma} (i\omega \rho v_n^i + p^i \frac{\partial g}{\partial n}) d\Gamma + p^i . \quad (18) \]

It is clear that e.g. in case \( P \in \Omega \)
\[
p_{s} = - \int_{\Gamma} (i \omega \rho v_{n} g + p \frac{\partial g}{\partial n}) d\Gamma ,
\]

or more precise
\[
p_{s} = - \int_{\Gamma} (i \omega \rho v_{n} g + (p' + p_{s}') \frac{\partial g}{\partial n}) d\Gamma ,
\]

\[
P_{s} = - \int_{\Gamma} (i \omega \rho v_{n} g + p s \frac{\partial g}{\partial n}) d\Gamma
\]

Because of equation (17), first part, a comparison with equation (15) gives us directly
\[
p' = - \int_{\Gamma} (i \omega \rho v_{n} g + p' \frac{\partial g}{\partial n}) d\Gamma = 0 .
\]

Notice that the incident wave does not satisfy the Sommerfeld radiation condition like the total and the scattered sound field do. The incident wave satisfies the inhomogeneous Helmholtz-equation in domain \( \Omega \) without the scatterer, resp. \( \Omega \cup S \), whereby the scattered sound field satisfies the Helmholtz-equation in the domain \( \Omega \):

\[
p'_{i} + k^{2} p'_{i} = - q \quad |_{\Omega \cup S} ,
\]

\[
p_{i} + k^{2} p_{i} = 0 \quad |_{\Omega} ,
\]

and this results in
\[
(p' + p_{s}')_{i} + k^{2} (p' + p_{s}') = - q \quad |_{\Omega}
\]

and with the first part of equation (17) one obtain again equation (6).

### 2.2 The BEM-approach for fluid-structure interaction

For fluid-structure interaction the Somigliana-identity, obtained from the Lamé-Navier-equation
\[
(c_{1}^{2} - c_{2}^{2}) u_{j,i} + c_{2}^{2} u_{i,j} + i \omega^{2} u_{i} = 0
\]

with \( c_{1}, c_{2} \) as the shear and dilatational speed of sound and \( u_{i} \) as the displacement vector, can be written as
\[
C_{y} u_{i}(P) = \int_{\Gamma} (u_{y} t_{j}) d\Gamma - \int_{\Gamma} (\tilde{t}_{y} u_{j}) d\Gamma ,
\]

with boundary traction vector \( t_{j} \) and the fundamental solution of the Lamé-Navier-equation \( \tilde{u}_{y} \)

\[
\tilde{u}_{y} = \frac{1}{4 \pi i} (\Psi \delta_{y} - \chi r_{y} r_{j}) .
\]

with shear modulus \( \mu \). The functions \( \Psi \) and \( \chi \) depend on \( r, k_{1}, k_{2} \) and can be found e.g. in [4].

Let us now assume that the scatterer itself is an elastic body. The corresponding equations for the coupling procedure are

\[
\begin{bmatrix}
H'_{A} & H_{K}'
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_{A}^{1}
\mathbf{u}_{K}^{1}
\end{bmatrix} =
\begin{bmatrix}
G_{A}^{1} & G_{K}^{1}
\end{bmatrix}
\begin{bmatrix}
\mathbf{t}_{A}^{1}
\mathbf{t}_{K}^{1}
\end{bmatrix}
\]

\[
\begin{bmatrix}
F'_{A} & F_{K}'
\end{bmatrix}
\begin{bmatrix}
\mathbf{p}_{A}^{1}
\mathbf{p}_{K}^{1}
\end{bmatrix} =
\begin{bmatrix}
J_{A}^{1} & J_{K}^{1}
\end{bmatrix}
\begin{bmatrix}
\mathbf{v}_{nA}^{1}
\mathbf{v}_{nK}^{1}
\end{bmatrix} +
\begin{bmatrix}
p_{A}^{\text{inc}}
p_{K}^{\text{inc}}
\end{bmatrix}
\]

where \( A \) indicates the non-coupling part of the boundary and \( K \) the coupling part, resp. the actual interface. To obtain steadiness in the coupling interface \( \Gamma_{K} \) we demand

\[
t_{K}^{1} = - p_{K}^{1} n_{K} - p_{K}^{1} n_{K}
\]

\[
i \omega u_{K}^{1} = v_{nK}^{2}
\]

where \( n_{K} \) is the normal vector in the coupling part of the boundary. The coupled system of equations can be written as, see NOLTE [4],

\[
\begin{bmatrix}
H'_{A} & H_{K}' & 0 & + G_{A}^{1} n_{K}
0 & - i \omega J_{K}^{1} n_{K}
F'_{A} & F_{K}' & 0 & + G_{K}^{1} n_{K}
0 & - i \omega J_{K}^{1} n_{K}
\end{bmatrix}
\begin{bmatrix}
\mathbf{u}_{A}^{1}
\mathbf{u}_{K}^{1}
\mathbf{p}_{A}^{1}
\mathbf{p}_{K}^{1}
\end{bmatrix} =
\begin{bmatrix}
G_{A}^{1} & O & t_{A}^{1}
0 & O & v_{nA}^{2}
\end{bmatrix}
\begin{bmatrix}
\mathbf{t}_{A}^{1}
\mathbf{v}_{nA}^{1}
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
p_{A}^{\text{inc}}
p_{K}^{\text{inc}}
\end{bmatrix}
\]

The dimensions of the submatrices are

\[
H'_{A}, H_{K}', G_{A}^{1} \cong 3 \times 3
\]

\[
F'_{A}, F_{K}', J_{K}^{1} \cong 1 \times 1
\]

\[
- i \omega J_{K}^{1} n_{K} \cong 1 \times 1
\]

\[
+ G_{K}^{1} n_{K} \cong 3 \times 1
\]

The coupling terms \( (p_{A}, u_{K}) \) are located on the left hand side of equation (30), they are defaultly unknown values. One can only influence the coupling values by the incident vector in (30).
It should be mentioned that only the Dirichlet data are calculated in the interface.
To obtain the Neumann data in the interface equation (29) can be used again. After all this a field point calculation for each domain can be done.
In figure 1 the entire boundary is the interface so that a non-coupling boundary does not exist in this special case (terms indicate with an $A$ in equations (28) and (30)).

2.3 A Kirchhoff-approach of first order

In the higher frequency range a BEM-approach failed because of limited computer memory. Remember at first one has to solve a system of equation on the boundary $\Gamma$ and afterwards a so called field point calculation can be done in the domain $\Omega$.

Moreover in the higher frequency range one can accept some more assumption to avoid the explicit process of solving a system of equations. And this leads to an approach which is comparable with a BEM-field point calculation. The first assumption is that the boundary $\Gamma$ is divided into a part which is illuminated $\Gamma_{ill}$ and another which is non-illuminated $\Gamma_{non-ill}$ by the source or in the bistatic case resp. by the receiver. The boundary integration procedure is now reduced on the illuminated part only, figure 2. Another assumption is a reflection coefficient $R$ on the boundary $\Gamma_{ill}$, so that the scattering sound field can be written as

$$p^s = Rp^i$$ \hspace{1cm} (32)

The gradient in normal direction on this boundary is

$$\frac{\partial p^s}{\partial n} = -R \frac{\partial p^i}{\partial n}$$ \hspace{1cm} (33)

This leads to the integral for the scattered field, because of (17) and (19),

$$p^s(P) = \int_{\Gamma_{ill}} \left[ (1 - R)g \frac{\partial p^i}{\partial n} - (1 + R)p^i \frac{\partial g}{\partial n} \right] d\Gamma$$ \hspace{1cm} (34)

It has already been shown that the integral over the incident wave for a point outside the scatterer - in the absence of the scatterer - vanishes

$$\int_{\Gamma_{ill}} g \frac{\partial p^i}{\partial n} + p^i \frac{\partial g}{\partial n} d\Gamma = 0 \hspace{1cm} \text{.}$$ (35)

So that equation (34) results in

$$p^s(P) = -\int_{\Gamma_{ill}} R \left[ g \frac{\partial p^i}{\partial n} + p^i \frac{\partial g}{\partial n} \right] d\Gamma$$ \hspace{1cm} (36)

E.g. in the case of a plane incident wave of intensity $I = 1$$

$$p^i(P) = I e^{-ikr}$$ \hspace{1cm} (37)

this results for $kr << 1$ in the integral expression for the scattering sound field

$$p^s(P) = -\frac{ik}{4\pi} \int_{\Gamma_{ill}} R e^{ikr} p^i \left[ \cos(\Phi^i) + \cos(\Phi^s) \right] d\Gamma$$ \hspace{1cm} (38)

depending on the angular $\left(\Phi^i, \Phi^s\right)$. The angulars are depicted in figure 2 and figure 3 as well. A short discussion of all these assumption should be done. To be consistent with equations (32) and (33) the incident wave is assumed to be zero on the non-illuminated part on the boundary. Note, that with this approximation the
integral (35) will not vanish and consequently the scattered field due to (34) will also not vanish in the limit of a vanishing reflection coefficient. The scattered field for the monostatic case, where the location of the load and the receiver point are coincident, is obtained from equation (38) with \( \cos(\Phi') = \cos(\Phi) \). In this case the argument of the integrand in equation (35) vanishes and this yields the entire integral to zero.

### 2.4 A Kirchhoff-approach of second order

The Kirchhoff approach of first order take - with some assumptions – the main diagonal terms of equation (14) into account. To improve the Kirchhoff-approach of the first order double scattering on some elements are considered, see figure 4. This is important in case the structure contains concave parts on the surface and in the case of a pure convex surface this can be neglected. In figure 5 another test structure is depicted.

#### Figure 4: Corner reflector

#### Figure 5: Test structure

### 2.5 A Kirchhoff-approach of order n

Because we already derived a Kirchhoff-approach for the first and second order we consequently can move straight forward to receive a Kirchhoff-approach of an arbitrarily order \( n \). Convergence could get lost with a certain order \( n_c \).

### 2.6 Target Strength definition

For a steady state signal the monostatic (fig. 2) target strength is defined as

\[
TS = 10 \log_{10} \frac{\left| p_{\text{sc}} \right|^2}{p_{\text{inc}}^2} = 10 \log_{10} \left( \frac{r^2}{r_0^2} \right), \quad (40)
\]

with the scattered acoustic sound pressure level at location \( r \) and \( r' \) at the center of the target. This is evaluated in the far field, \( \lambda / a << 1 \), where \( a \) denotes the characteristic length of the target or obstacle. In the far field TS is not depending on \( r \).

### 3 Some generic results

#### 3.1 The BEM approach for a spherical scatterer

The BEM-approach for a classical rigid sphere is calculated. In figure 6 the incident pressure field is depicted. The scattered field in a plane perpendicular to the incident wave is shown in figure 7.

#### Figure 6: Incident pressure field (N/m²)
4 Summary

In this paper different approaches for scattering phenomena were developed. The exact solution is a BEM-approach which takes the fluid-structure interaction into account. One obtains such an approach by a BEM-BEM coupling method; BEM-approach is used for the fluid part as well as for the structure part. The advantage of this method occurs when two or more domains of infinite sizes interact with each other including sources in the fluid domain. Some Kirchhoff approaches were discussed as well.

References