The observations of infra-sonic propagation of impulsive (thunder, explosions, or supersonic planes) distant sources show an "abnormal" duration and chaotic oscillations. We call this phenomenon "rumble" and try to explain it. Assuming Navier & Stokes equations are an undeniable model, we describe the chain of hypothesis and assumptions which lead to the wave front propagation (rays tracing): nonlinear effects, dissipative processes and geometric acoustics. As this last model cannot explain the rumble (an impulsive source must give an impulsive response), we discuss the previous hypothesis. At first analysis, neither non linear effects nor dissipative processes can yield such a phenomenon. The best candidate seems to be the diffractive effects (so geometric acoustics assumption is not valid). Assuming the atmosphere is a stratified medium due to the gravity, we are able to compute the atmospheric transfer function in the case of a constant sound velocity. A dimensionless number naturally appears ($r.g/a^2$ distance times gravity divided by the square of the sound speed) and when it is greater than 1, the impulse response shows long duration and oscillations.

1 Introduction

The observations of infra-sonic propagation of impulsive distant sources show a strange behavior. In the first section we describe the main characteristics of this phenomenon which we call "rumble": lengthening of the duration increasing with the distance, chaotic fluctuations at very low frequencies not modifying the spectrum between 1 and 10 Hz, azimuth, elevation angles and travel times are in conformity with a traditional calculation by the method of ray tracing. To find the causes, we analyze the different hypothesis for the physical model: Navier & Stokes equations, linearized Navier & Stokes equations, linearized Euler equations and geometric acoustics. We try to explain through the notion of transfer function of the atmosphere why the wave front propagation cannot explain the rumble. We also show that the sound propagation of a supersonic source is very similar to the fixed source. Then we discuss the non-linear effects and dissipative processes. They cause a lengthening of the duration, but at first analysis, neither non linear effects nor dissipative processes can explain the "rumble". So only the diffractive effects remain. In fact, because of the gravity, the atmosphere is a stratified medium, i.e. this medium is not homogeneous and the wave equation must be modified, even in the case of constant sound velocity. If this assumption is made (this is not physically meaningless) we are able to compute the atmospheric transfer function and a dimensionless number naturally appears ($r.g/a^2$ distance times gravity divided by the square of the sound speed and when it is greater than 1 (when the distance is greater than 10 km which is compatible with the observations) the transfer function is different of that in the homogeneous case. In addition we can compute the impulse response and it shows long duration and oscillations. At first sight this behavior is similar to the observations, but must be completed with more sophisticated computations in order to be more realistic.

2 The problem of the "rumble" of distant sources

To explain what we understand by rumble of remote source, we will take the example of a storm. The sonic source at the origin of the thunder is a flash of duration of few seconds. If the source is close to the listener (3 km or 10 seconds of propagation) the received signal last few seconds. On the other hand if the source is farther (20 or 30 km), one can hear far away a "rumble" which can last until several minutes. The same phenomenon was observed in a systematic way during the First World War where the noise of the enemy batteries thundered far off. The recordings of the experiment of Misty Picture [1] show this phenomenon: although the source (a chemical explosion) is very impulsive, the recording of the infrasonic vibrations at long-haul shows one "abnormal" duration of several minutes with a little chaotic oscillations of long period, like a tremor. For an infrasonic source due to supersonic aircraft type (Concorde for instance, [4]) one can observe the same phenomenon: under the trajectory, one can observe a traditional "N-wave" which lasts one or two seconds, longer than the wave near the source. In the Concorde case, the plane is 60 m length and the speed of Mach 2 ($\simeq 600m/s$) thus a duration about 0,1 s. At an intermediary distance one can observe after the N-wave a kind of wake. The farther is the source, the longer is the received signal. The directional analysis shows that the site and the azimuth of the wave remain appreciably constant (see [7]. And the spectral analysis between 1 and 10 hertz does not show variation of the spectrum . Moreover the travel time, the azimuth and the site can be worked out rather well [2]. Thus "rumble" seems to be characterized by:

- "abnormal" lengthening of the duration of the signal increasing with the distance,
- chaotic fluctuations at very low frequencies not modifying the spectrum between 1 and 10 Hz.
- azimuth, elevation angles and travel times are in conformity with a traditional calculation of the method of ray tracing.

3 Physical models: from the Navier & Stokes equations to the rays tracing

3.1 Chain of hypothesis

The details can be found in [2]. For the study of the propagation of the sound in the atmosphere, an undeniable model are the equations of Navier & Stockes: these equations express the balance of mass, momentum, and energy (to take into account the possible phenomena of molecular vibration, one can possibly consider several equations of energy but it is nevertheless a refinement of the theory). In the terms of "fluxes", one can distinguish the terms of the hyperbolic type (or "Euler") of the dissipative terms (viscosities and thermal conduction) in general weak. As the fluctuations of pressure relating to the propagation of the sound are few Pascals and the pressure at rest is $10^5 P$, the first simplification consists in linearizing the equations and to reduce them to the *equations of linearized Navier and Stockes (LNSE)*. This assumption, discussed in section 4 , thus consists in neglecting the non-linear effects. The second assumption consists to neglect the viscous effects and to be reduced to the equations of *linearized Euler equations (LEE)* . These (linear) equations, are naturally written at the order 1 (i.e. the principal symbol is a matrix valued symbol of order one, the symbol of the convected wave equations) but a symbol of order zero (or a multiplication term) is present too. As in this term only the gradients of the basic flow or the medium intervene, it disappears when the medium is homogeneous. The last assumption made in general, is the assumption of geometric acoustics which gives the equation of "eikonal" of which resolution utilizes the bicaracteristic curves (or rays). So we have a chain of hypothesis:

- nonlinear effects
- dissipative processes
- geometric acoustics

3.2 Notion of the transfer function of the atmosphere

If we assume an homogeneous open medium and an harmonic source, the sound propagation is described by the well known Helmholtz equation. If we interpret the atmosphere as a linear filter, the Green function as function of the frequency (the distance being considered as a characteristic parameter of this filter) is nothing but the transfer function of the atmosphere. In the usual (3-D case) propagation this function is:

$$exp(ikr)/r$$

where $k = 2\pi/\lambda = 2\pi f/a$, r the source-receiver distance and a the sound velocity. We notice:

- the term $1/r$ is the attenuation of the filter (pure gain)
- the term $exp(ikr)$ is a pure retard due to the propagation time.

For a supersonic slend body moving (along the horizontal x-axis) at constant speed in a homogeneous open medium, the propagation of the fluctuations of the pressure can be described in the (moving) frame of the source by the wave equation where the x-axis plays the role of the time and the sound velocity is the inverse of the Glauert term ($1/\beta = 1/\sqrt{M^2 - 1}$, M Mach number) :

$$(M^2 - 1)\partial^2_{xx} p' - \Delta_2 p' = 0$$

and the source is connected with the shape of the body and is usually impulsive. (see [3]). The pressure signal at a *vertical* distance r can be obtained by convolution of the source by the impulse response of the medium which is in this case 2-dimensionnal fundamental solution. In theory, there is a deep difference between odd or even space dimension for the wave equation. In odd case (usual 3-d case, e.g.) the fundamental solution (Green "function") is a distribution whose support is the sphere of radius $r = at$ whereas in the even case the support is the cone $t \geq r/a$: for instance in the supersonic case the fundamental solution is:

$$1/2\pi\sqrt{x^2 - \beta^2 r^2}$$

This is important for us because that means the response of an impulsive source is in 2-D case not impulsive. In effect the support of the convolution is the whole cone, which seems at the first sight completely different of the 3-D case. In fact the 2-D and the 3-D propagations are very similar. If we want to compute the 2-D transfer function, we have to solve the 2-D Helmholtz equation whose solution is the Hankel function $\sqrt{r} H^1_0(kr)$. Its asymptotic behavior for far propagation is (apart from numerical factor):

$$H^1_0(kr) \simeq e^{ikr}/\sqrt{kr}$$

And as in the 3-D case, one has for the atmospheric transfer function (let us say again that r is the *vertical* distance)

- the term $1/\sqrt{r}$ is the attenuation of the filter (pure gain) now cylindrical attenuation
- the term $exp(ikr)$ is a pure retard due to the propagation time.
- only the term $\sqrt{k}$ makes the support of the fundamental function not be a point.

In the case of the geometric acoustics hypothesis, when the medium is slowly heterogeneous, what is usually done gives in a certain sense the same results: the distance is replaced by the travel time along the rays and the stretching of the rays are taken into account. One obtains new (with respect of homogeneous case) phenomena like multiple arrivals, caustics... but the response to an impulsive signal remains impulsive. If we want to explain our rumble we have to analyze the three previous hypothesis (assuming the Navier & Stokes equations are valid).

4 Nonlinear effects and dissipative processes

We discuss briefly the too first hypothesis (see [3] pp 587). These effects are rather complicated and we cannot here get into the matter more rigorously.

4.1 Non linear effects

As non linear, these effects do not exist for weak sources. In addition, for long propagation the "geometric" decreasing (in $1/r$ for 3-D case or $1/\sqrt{r}$ for 2-D case) makes the non linear effects relevant only near the source at the beginning of the propagation. For instance in the case of the supersonic plane Concorde these effects are no more present at the boundary of the primary carpet (about $40 km$) where the N-wave is smoothed by the dissipation. But the signal lasts 2 or 3 seconds which must be compared with the initial duration of $0.2s$ (in fact a bit longer due to the wake). This is well explained and described by the Burger equation: the velocities of the front and tail discontinuities make the duration of the N-wave longer. But in this case, they cannot explain a duration of more than few seconds.

4.2 Dissipative processes

Although the *Navier-Stokes-Fourier model* is not in accord with the experiment (in the atmosphere), it gives a qualitative behavior of the dissipative processes. The transfer function of this simplest model is:

$$F(f) = exp(-\alpha r f^2)$$

If the distance of propagation r is small, the transfer function is equal to 1 and the filter has no effect. Otherwise, the filter destroys the high frequencies, only the infra-sound are propagated. The main effect on the signal is a " smoothing effect", the signal is "spread" : for the N-wave the rise-time is increased and yields a longer duration. At first analysis, neither non linear effects nor dissipative processes can explain the "rumble".

5 Diffraction of a medium stratified by gravity

Now, we shall establish the *stratified Helmholz* equation in the case of a stratified medium without wind; the most general case can be found e.g. in [2]. It is a special case of L.E.E. The index 0 stands for the quantities of the medium (f_0 for the quantity f). The steady-state solution of the Euler equation for the medium is given:

- by the z-axis momentum equation: (z is the vertical axis)
$$\partial_z P_0 = -\rho_0 g$$
P_0 is the pressure, ρ_0 the density and g the gravity.
- by a given temperature profile $T_0 = T(z)$ (depending on the flux of energy) and the law of perfect gazes:
$$P = \rho RT$$

The profiles $P_0(z)$ and $\rho_0(z)$ are obtained by a straightforward computation. For the fluctuations (with index 1, f_1 for quantity f) ,it is convenient to introduce:

$$\mu = \frac{\rho_1}{\rho_0(z)}$$

If we assume, as usual, isentropic transformations one has:

$$dp_1 = a^2 d\rho_1 \quad a : \text{sound speed}$$

One obtains for the momenta and the conservation of the mass:

$$\begin{aligned} \partial_0\, u_1 + a^2\partial_x\mu &= 0 \\ \partial_0\, v_1 + a^2\partial_y\mu &= 0 \\ \partial_0\, w_1 + a^2(\partial_z(\rho_0\mu)/\rho_0 + g\mu &= 0 \\ \partial_0\, \mu + \partial_x u_1 + \partial_y v_1 + \partial_z w_1 + \rho_0'/\rho_0 w_1 &= 0 \end{aligned} \quad (1)$$

If we forgot the index 1 (no ambiguity) one obtains:

$$\begin{aligned} \partial_0\, u + a^2\partial_x\mu &= 0 \\ \partial_0\, v + a^2\partial_y\mu &= 0 \\ \partial_0\, w + a^2(\partial_z\mu + k_w\mu) &= 0 \\ \partial_0\, \mu + \partial_x u + \partial_y v + \partial_z w + k_\mu w &= 0 \end{aligned} \quad (2)$$

with for k_μ and k_w (their dimension is the inverse of a length):

$$\begin{aligned} k_\mu = \rho_0'/\rho_0 = P_0'/P_0 - T_0'/T_0 &= k_\theta - \gamma g/a^2 \\ k_w = g/a^2 + k_\mu &= (\gamma-1)g/a^2 + k_\theta \end{aligned} \quad (3)\ (4)$$

with $k_\theta = T_0'/T_0$ The order of magnitude of g/a^2 is $10^{-4} m^-1$. A very important remark for the sequel is: *the characteristic length of the gravity effects is 10 km.* For altitude under 11 km one has $1/k_\theta \simeq 30km$ in the I.S.A case. When the medium is homogeneous the equations 3 are the "wave equations" written at the order one. When the medium is heterogeneous (stratified) the "wave equations" written at the order one must be completed by *multiplication term*, i.e. a symbol of order zero. To take in account this correction, one can use the well-known *Born method* (see [3] pp 441-443). The multiplication terms are like secondary sources (see figure 1), their travel times are longer and the signal lasts longer. Moreover interferences appear.

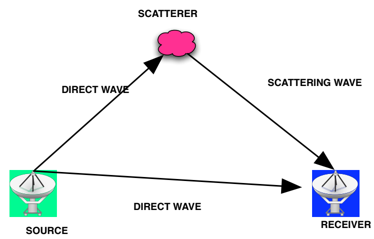

Figure 1: Born approximation scheme

To obtain qualitative behavior of the atmospheric transfer function (A.T.F.) we assume a constant sound velocity. This hypothesis is not unphysical : for instance the temperature of the I.S.A. is constant above 11 km or this hypothesis can happen above the ground depending on the meteorology. In this case the Green function $G(M, M_s)$ of the *stratified Helmholtz equation* verifies equation 5 (k is the wave number):

$$(k^2 + k_w k_\mu)G + (k_w + k_\mu)\partial z G + \Delta G = \delta(M - M_s) \quad (5)$$

If we write:

$$G = e^{bz} R$$

As (ϕ is a test function),

$$< e^{-bz}\delta, \phi >=< \delta, e^{-bz}\phi >= e^{-b0}\phi(0) = \phi(0)$$

R is solution of the *usual Helmholtz equation*, if we choose : $b = -(k_w + k_\mu)/2$.

$$G = e^{-(k_w - k_\mu)z} R$$

$$\text{and } (k^2 - k_0^2)R + \Delta R = \delta \quad (6)$$

$$\text{so } R = \frac{exp(-r\sqrt{k_0^2 - k^2})}{r} \quad (7)$$

with $k_0 = (k_w - k_\mu)/2$ and r is the distance between the source and the listener. When $k_0^2 \geq k^2$ the root is positive and the Green function satisfies the radiation condition when $r \to \infty$ and must be analytic otherwise. When k_0 is equal to zero, the solution is of course the free-space Green function. If we express k in term of frequencies, one obtains the A.T.F. One can notice:

- the gravity effects are not negligible when the dimensionless ratio $k_0.r$ becomes of order 1 i.e. $r \simeq 10km$
- one has an effect of the gain of the transfer function depending on the relative altitude of the source and the listener. As the medium is heterogeneous the Green function is not symmetric.
- the phase of the transfer function is no longer $f.r$, so we obtain a great change.

6 Atmospheric impulse response

To obtain the impulse response we have to compute the inverse Fourier transform of the transfer function. We have done it approximatively by the method of the stationary phase (when the ratio $k_0.r$ is great). We introduce the frequency in the Green function $k = 2\pi/\lambda = 2\pi f/a$. Idem for k_0. The impulse response is given by (with forget the amplitude depending on r):

$$S(t) = \int_{-\infty}^{-\infty} exp(+2\pi[ift - r/a\sqrt{f_0^2 - f^2})]df$$

The sign "-" has been chosen in the root to the wave be decreasing if $f < f_0$ and outgoing if $f > f_0$ when $r > 0 \to \infty$, so $\sqrt{f_0^2 - f^2}$ means $isgn(f)\sqrt{f^2 - f_0^2}$ if $f > f_0$.
If $f_0 = 0$ one obtains the retarded wave:

$$S(t) = \int_{-\infty}^{-\infty} exp[+2\pi if(t - r/a)]df = \delta(t - r/c)$$

(One has to check the transfer function must be the Fourier transform of a real signal.) If we set :

$$f = \tilde{f} f_0 \quad N = r/a.f_0 \quad t = \tau r/a$$

we obtain:

$$S(t) = f_0 \int_{|\tilde{f}|\geq 1} exp[+2i\pi N(\tau\tilde{f} - sgn(\tilde{f})\sqrt{\tilde{f}^2 - 1})]d\tilde{f} + \int_{|\tilde{f}|\leq 1}$$

If we assume $N \gg 1$, one can use the method of the stationary phase. A straightforward computation gives:

$$S(t) \simeq \frac{\cos(\pi/4 + N\sqrt{\tau^2 - 1})}{(\tau^2 - 1)^{3/4}} \frac{f_0}{\sqrt{N}}$$

The signal verifies the *Huygens principle* i.e. is equal to zero when $\tau = at/r < 1$. In addition, the function is integrable when $\tau \to 1$, the exponent being 3/4.

7 Conclusions and perspectives

To explain *the rumble phenomenon* (abnormal duration and oscillations at long distance of propagation) described in section 2 , we investigated different hypothesis:

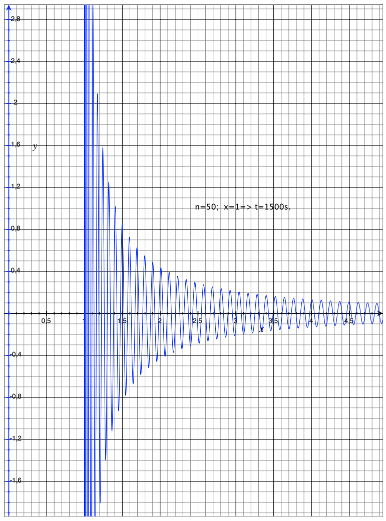

Figure 2: atmospheric impulse response at 50 km versus adimensionnal time ct/r

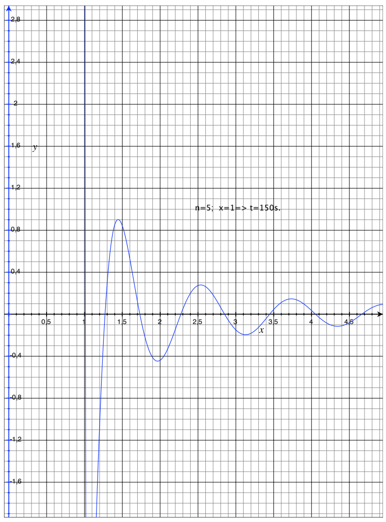

Figure 3: atmospheric impulse response at 5 km versus adimensionnal time ct/r

- Non-linear effects (section 4)
- Dissipative processes (section 4)
- Diffraction effects due to the gravity (section 5)

If the two first hypothesis cause a longer duration of an impulsive source, this duration cannot be sufficient to explain the observations at long distance. Otherwise the diffractive effects due to the gravity seem to be the best candidate. In the academic (but realistic) case of a constant sound velocity where the medium remains stratified, we obtained the transfer function of the atmosphere, which is nothing but the Green function of the *stratified Helmholtz equation*. This function, approximatively equal to the usual free-space Green function when an dimensionless constant ($k_0.r$ with $1/k_0 \simeq 10km$) is small has a dramatic different behavior when it becomes greater than 1, which seems to agree with observations. To be more realistic and to obtain multiple arrivals and caustics, it is necessary to take in account gradients of temperature, wind profiles, and boundary condition at the ground. To do that, one can envisage:

- Numerical simulations of L.E.E, but as the domain

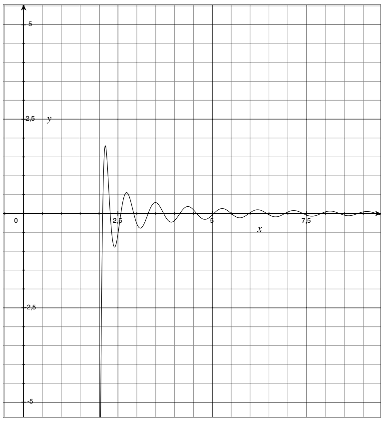

Figure 4: atmospheric impulse response after a travel time of 400 s versus time

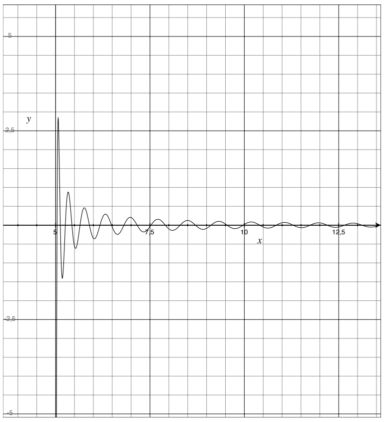

Figure 5: atmospheric impulse response after a travel time of 1000 s versus time

contains a important number of wavelength, they will be rapidly heavy.

- As the medium is 1-dimensional, one can obtain ordinary differential equations in z (instead of partial differential equation as in the previous point) with use of adapted Fourier transform.
- It is probably possible to compute the first term of the *WKB asymptotic development* which takes into account the multiplication term (symbol of order zero) (c.f. [8] or [9] for instance)

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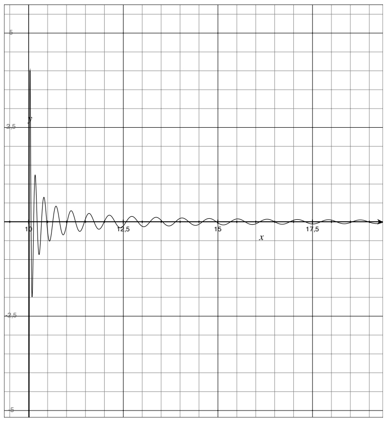

Figure 6: atmospheric impulse response after a travel time of 2000 s versus time

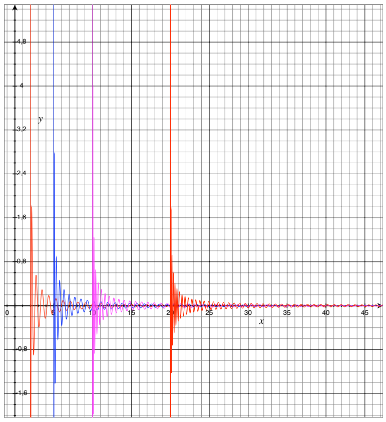

Figure 7: atmospheric impulse response after different travel times versus time

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