



On the generation of axial modes in the nonlinear vibration of strings

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Vibrating strings are known to be nonlinear. Transverse vibrations induce axial motion as well as a modulation of the string's tension. An overview of the nonlinear models of the vibrating string has been presented at the 150th Meeting of the ASA. It has been shown that if temporal considerations are prevalent from an acoustical point of view, the involved coupling mechanisms are mainly ruled by the spatial shape of the modes.

The purpose of this communication is to investigate the specific case of transverse-axial interactions and to examine the possible generation of axial modes through transverse vibrations. It also gives the opportunity to discuss the common hypotheses.

Despite the complex modal couplings that can occur, this results enable to avoid or favor axial modes in the design of the string itself or of the boundary conditions *i.e.* the bridge.

1 Introduction

In the classical problem of the vibrating string, intrinsic nonlinearities play a major role and still are a subject of considerable interest in the literature. Since the seminal paper of Carrier [1], a lot of models have been proposed, the more general seeming due to the author [2] who had the opportunity to present and discuss the various models and underlying hypotheses in [3].

The equations of motion lead to intricate couplings and a lot of particular situations have been studied, mostly in a single transverse mode case and formerly with a mainly temporal point of view.

The axial equation of motion appears to have the form of an inhomogeneous linear wave equation driven by the transverse motion, and on which we will focus in this communication.

For some applications, such as musical acoustics where the design of the bridge can take advantage of axial motion, it is valuable to clearly understand the underlying mechanisms of axial modes generation. Else, if as usually, axial motion is not the most relevant issue, it indirectly affects the transverse motion.

After having derived the equation and defined what an axial mode is, as well as insisted on the distinction with longitudinal motion, we will discuss the case of uniform extension, responsible for the octave stretch, and derive the corresponding solution in a more consistent way than usual [4].

Finally, the true nature of axial modes generation will be explained. It will be shown that it is mainly ruled by the spatial shape of the modes. The understanding of the involved mechanisms enables to avoid or favor axial modes.

2 Kinematics

We consider a string stretched at rest between its ends at $z = 0$ and $z = L_o$. A point of initial coordinates $(0, 0, z)$ in a three-dimensional orthonormal Cartesian coordinate system is displaced to $(x(z, t), y(z, t), z + u(z, t))$ as shown in Fig.1.

As well as x and y designate the *transverse* displacements, it should be pointed out that u describes an *axial* motion (displacements parallel to the string's axis z at rest), not to be confused with the *longitudinal* waves, tangent at any instant to the deformed shape of the string and characterized by the local extension $a(z, t)$.

Superimposed dots will denote derivations with respect to time t and primes derivations with respect to z .

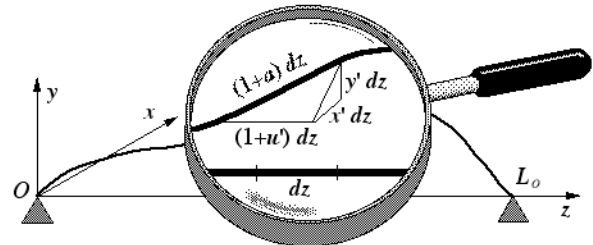


Fig.1 Kinematics.

The actual length ds of an elementary portion of the string of length dz at rest is given by

$$ds = (1 + a) dz = \sqrt{(1 + u')^2 + x'^2 + y'^2} dz \quad (1)$$

and the unit vector tangent to the string writes

$$\begin{aligned} \vec{t} &= \frac{dz}{ds} [(1 + u')\vec{z} + x'\vec{x} + y'\vec{y}] \\ &= \frac{1}{1 + a} [(1 + u')\vec{z} + x'\vec{x} + y'\vec{y}] \end{aligned} \quad (2)$$

3 The axial equation of motion

3.1 Exact equation of motion

For a perfectly flexible string of lineic mass density at rest $\mu_o (= \rho_o A$, where ρ_o is the volumic mass density, for an homogeneous cylindrical rod), the exact equation for free motion writes

$$\frac{\partial}{\partial z} \vec{T} = \mu_o \vec{y} \quad (3)$$

where

$$\vec{T}(z, t) = T(z, t) \vec{t}(z, t) \quad (4)$$

is the local actual tension.

Projection of Eq.(3) on the z axis, gives:

$$\frac{\partial}{\partial z} \left(\frac{1 + u'}{1 + a} T \right) = \mu_o \ddot{u} \quad (5)$$

3.2 Elastic behaviour

Assuming the string to be linearly elastic in the vicinity of the prestressed state with tension $T_o (> 0)$ at rest, the value of the local actual tension takes the form

$$T(z, t) = T_o + K a(z, t) \quad (6)$$

where K is a constant (for an homogeneous cylindrical rod, $K = EA$, where E is the tangent Young's modulus and A the area of the cross sections).

3.3 The approximate axial equation of motion

Retaining only terms of the order of a , Eq.(5) becomes

$$\ddot{u} - c_L^2 u'' = \frac{1}{2} (c_L^2 - c_T^2) (x'^2 + y'^2)' \quad (7)$$

where

$$c_L^2 = \frac{K}{\mu_o} \quad \text{and} \quad c_T^2 = \frac{T_o}{\mu_o} \quad (8)$$

are the axial and transverse wave velocities in the ideal string case.

The well known Eq.(7) [2] appears to have the form of a *linear wave equation forced by the transverse motion*.

This forcing only disappears in the particular situation where $c_L = c_T$, corresponding to a tension-length proportionality, in which case Eq.(3) reduces to the linear string equation, whatever the amplitude may be [5].

In practice

$$c_L \gg c_T \quad (9)$$

3.4 Form of the solution

For fixed ends ($u(0) = u(L_o) = 0$), the axial motion can always be written under the form

$$u(z, t) = \sum_n u_n(t) \sin \frac{n\pi z}{L_o} \quad (10)$$

with

$$u_n(t) = U_n(t) \cos(\Omega_n t) \quad (11)$$

and

$$\Omega_n = \frac{n\pi}{L_o} c_L = n \Omega_o \quad (12)$$

Each term in the sum of Eq.(10), with Eqs.(11) and (12), defines an axial mode.

4 On the transverse modes

4.1 Approximate transverse modes

For pinned ends, the transverse motion can be written under the same form as Eq.(10) but with non harmonic temporal parts. Nevertheless, the fundamental frequency ω_m of the m -th transverse mode is slightly greater than that of the ideal linear case

$$\omega_m = \frac{m\pi}{L_o} c_T = m \omega_o \quad (13)$$

the detuning being due to the increase of the mean tension resulting from the string's extension during the motion. The neglected higher order components are corrections of very small amplitudes with respect to those of the fundamentals.

Moreover, the transverse equations of motion lead to awkward couplings [3], but we can suppose that their

amplitudes slowly change with time and finally assume for x and y temporal parts of the form of Eq.(11).

4.2 On the flexural and torsional stiffnesses

We have derived the axial equation of motion in its simplest way. Consideration of flexural and torsional stiffnesses only affects the transverse equations of motion and is of poor interest here [2]. The most important indirect influence is that flexural stiffness leads to another slight increase of the transverse modes frequencies.

5 Uniform extension

A transverse vibration implies a modulated increase of the actual string length given by

$$L(t) = \int_0^{L_o} (1+a) dz = L_o + \frac{1}{2} \int_0^{L_o} (x'^2 + y'^2) dz \quad (14)$$

where we have considered a constant distance between the ends:

$$\int_0^{L_o} u' dz = 0 \quad (15)$$

Eqs.(1) and (7) gives the local extension

$$\begin{aligned} a'(z, t) &= u'' + \frac{1}{2} (x'^2 + y'^2)' \\ &= \frac{1}{c_L^2} \left[\ddot{u} + \frac{1}{2} c_T^2 (x'^2 + y'^2)' \right] \end{aligned} \quad (16)$$

Assuming that Rel.(9) holds, one can consider that the present frequencies are too low to excite longitudinal waves and that the extension a is thus uniform along the string, which writes

$$a_{unif}(t) = \frac{L - L_o}{L_o} = \frac{1}{2L_o} \int_0^{L_o} (x'^2 + y'^2) dz \quad (17)$$

The extension being directly related with the tension by Eq.(6), this leads to what is called the *octave stretch* at twice the frequencies of the transverse modes [4]. This phenomenon can be exhibited in the Raman experiment where an end of the string is attached to the center a perpendicular plate driven by the string's tension.

The corresponding axial displacements, given by Eq.(1) or Eq.(16) with $a' = 0$, are

$$\begin{aligned} u_{unif}(z, t) &= \frac{z}{2L_o} \int_0^{L_o} (x'^2 + y'^2) dz - \frac{1}{2} \int_0^z (x'^2 + y'^2) dz \end{aligned} \quad (18)$$

with

$$\ddot{u}_{unif}(z, t) = -\frac{1}{2} c_T^2 (x'^2 + y'^2)' \quad (19)$$

which is not zero, accordingly to an oscillatory motion.

The method followed here is more consistent than the classical one [4, 6] where c_T is neglected with respect to c_L and assuming that transverse frequencies are too low to excite *axial modes*, u_{unif} is calculated from Eq.(7) in the

quasi-static case ($\ddot{u} = 0$). It gives the same result as Eq.(18), but such an *a posteriori* procedure which only holds in the limiting case $c_L / c_T \rightarrow \infty$ increases the non-linearity of the transverse equations of motion and can be incompatible with the orders of magnitude of the terms retained in the transverse and axial equations of motion.

Moreover, the resulting axial motion do not correspond to axial modes since the spatial and temporal parts, imposed by the transverse motion, are not related to the same axial wave number (see also [7] where the first axial mode appears only because initially present).

6 Generation of axial modes

6.1 Necessary condition

Injecting solutions of the form of Eq.(10) for the transverse and axial displacements in the equation of motion Eq.(7), and following a Galerkin procedure yields the system of axial modal equations [3]

$$\ddot{u}_n + \left(\frac{n\pi}{L_o}\right)^2 c_L^2 u_n = -\frac{1}{4} (c_L^2 - c_T^2) \left(\frac{\pi}{L_o}\right)^3 \times \sum_{i,j} \{nij (x_i x_j + y_i y_j) [\delta(i-j\pm n) + \delta(i+j-n)]\} \quad (20)$$

for each n , and where δ is the Kronecker symbol.

The right hand side exhibits selecting rules concerning the ranks of the modes, *i.e.* their spatial distributions, which are *necessary conditions* to excite axial modes. Thus, the n -th axial mode is excited *if and only if* are present two different transverse modes, of ranks i and j and same polarization, such that for $i \neq j$:

$$j \pm i = n \quad (21)$$

or one transverse mode such that:

$$2i = n \quad (22)$$

6.2 Efficiency condition

We see in Eq.(20) that, without loss of generality, we can only consider transverse modes in one polarization. Substitution of the approximate form discussed in § 4 yields

$$\ddot{u}_n + \left(\frac{n\pi}{L_o}\right)^2 c_L^2 u_n = -\frac{1}{8} (c_L^2 - c_T^2) \left(\frac{\pi}{L_o}\right)^3 \times \sum_{i,j} \{nij X_i X_j [\cos(i+j)\omega_o t + \cos(j-i)\omega_o t] \times [\delta(i-j\pm n) + \delta(i+j-n)]\} \quad (23)$$

and the n -th axial mode is forced at the frequencies present in the right hand side which must be near Ω_n . It appears immediately that conditions (9) and (22) cannot be satisfied together so that axial modes generation cannot occur in a single mode transverse vibration [8].

Rel.(9) with Eq.(21) show that an efficient excitation of the n -th axial mode can take place if

$$j - i = n \quad \text{and} \quad j + i \approx n \frac{c_L}{c_T} \quad (24a, b)$$

As an example, for a given string, it is quite easy to generate the first axial mode with two consecutive

transverse modes verifying Rel.(24b), and to avoid this excitation by suppressing one of them (for example by plucking the string at one of its nodes).

Eq.(23) shows that, if excited at its natural frequency, the axial mode's amplitude will grow linearly.

These conditions explain the experimental results obtained by Cuesta and Valette [6, 9].

6.3 Remarks

This investigation is useful for the transient part of the vibration. Once an axial mode is present, it will itself interact with transverse modes and modify the actual situation, leading to a more complex motion.

7 Conclusion

The purpose of this communication was to study the axial motion in a vibrating string. The classical result in the case of an uniform extension was obtained in a rigorous way and the axial modes generation mechanisms, obeying simple selecting rules based on the spatial modes shapes. This gives a tool to avoid or favor axial modes.

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