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Wave equation in non-integer-dimensional porous media

Claude Depollier and Jean-Baptiste Legland

Laboratoire d'Acoustique de l'Université du Maine, Avenue Olivier Messiaen, 72085 Le Mans,
France

claude.depollier@univ-lemans.fr

The present study deals with the sound wave propagation in fractional dimensional porous media. Using the Stillinger-Palmer-Stavrinou formalism, we establish the generalized Biot's wave equations for sound waves propagating in saturated porous media. As an application we study the wave equations and their solutions in a porous media having anisotropic fractional dimensions.

1 Introduction

Fractional dimension of space is now a well accepted tool to describe effective parameters of physical systems. The most famous exemple of fractional dimension is met in fractal geometry. A fractal is a quantity which displays self similarity on all scales. In physics, behind this word, we understand object or phenomena which cannot be decribed by smooth functions. One of the fascinating interest of fractals is their capability to model objects with complicated structures. This is due to an important property of the fractals objects that their structure is characterized by a small number of parameters. One of them is the fractal dimension which tells how the fractal is confined in the Euclidean space or how it fills the space in which it lies.

Very early, fractal concept has been incorporated in the study of porous media [1] to investigate various phenomena as flows in porous media pores and fractures. When a fluid moves in a porous medium, what happens in a part of the fluid is affected by the motion of the other parts. This fact in particularly salient in a fractal medium where the hierarchy of scales needs new equations of motion.

In [2] a formalism is developed by Stillinger to construct a generalized Laplacian operator which is convenient for its extension to non integer dimensional spaces. For a 2-spatial coordinates space, the Stillinger's formalism shows that it is possible to distribute the D dimensions between them. More recently, Palmer and Stavrinou generalized the results of Stillinger to n orthogonal coordinates [3]. One conclusion of their investigation is that this formalism allows to describe an anisotropic confinement of a fractal medium, *i.e.* different degrees of confinement can be associated to each orthogonal direction. They also derive the equation of motion in a non integer dimensional space.

The paper is organized as follows. In section 2 the Stillinger's formalism and the Palmer and Stavrinou derivation are briefly summarized. Section 3 is devoted to the Lagrangian formulation of the acoustical waves propagation in porous media leading to the Biot's equations. Lastly, in section 4, we derive the Biot's wave equations which occure in a fractional dimensional space.

2 Integration in spaces with non-integer dimension

In his paper [2] Stillinger developped a formalism which allows to write the Laplace operator in spaces having a fractional dimension D . What means a fractional dimensional space? A such notion is defined from the integral calculus as follows. Let us consider the integration of a radially symmetric function f , in a D -dimension

space

$$\int d\mathbf{x}_0 f(r(\mathbf{x}_0, \mathbf{x}_1)) = \int_0^\infty dr v_1(r) f(r) \quad (1)$$

where $r(\mathbf{x}_0, \mathbf{x}_1)$ is the distance between points \mathbf{x}_0 and \mathbf{x}_1 . Here,

$$v_1(r) = \sigma(D) r^{D-1} \quad (2)$$

and

$$\sigma(D) = \frac{2\pi^{D/2}}{\Gamma(D/2)}. \quad (3)$$

When D is an integer, $\sigma(D)$ agrees with the volume of the unit sphere in Euclidean spaces. This justifies the generalization of fractional dimension to any value of D . With this formalism, Stillinger shows that the Laplace operator in a D -dimensional space is

$$\nabla^2 f(r) = f''(r) + \frac{D-1}{r} f'(r) \quad (4)$$

For a non integer D -dimensional space, the Stillinger's formalism leads to a Laplace operator for which the non integer dimension is located only in one direction. For example, in a space where only the dimension of the p coordinate is integer, the Laplacian becomes :

$$\nabla^2 f(p, l) = \left[\frac{\partial^2}{\partial p^2} + \frac{\partial^2}{\partial l^2} + \frac{D-2}{l} \frac{\partial}{\partial l} \right] f(p, l) \quad (5)$$

3 Euler Lagrange equations in noninteger dimension

C. Palmer and P.N. Stavrinou[2], have generalised the Stillinger's formalism of noninteger dimensionnal spaces to n orthogonal coordinates. In this framework, and using the variational principle, they derive the Euler Lagrange equations of a field theory in such spaces which follow from the stationnarity property of the action integral with respect to variations of the fields and their derivatives. So, if the action is defined by

$$S = \int \mathcal{L} dv dt, \quad (6)$$

where $\mathcal{L} = \mathcal{L}(\phi_k, \partial_\mu \phi_k)$ is the Lagrangian density corresponding to a definite point of the space-time, the Euler-Lagrange equations are:

$$\frac{\partial \mathcal{L}(\phi_i, \partial_\mu \phi_i)}{\partial \phi_i} - \partial_\mu \frac{\partial \mathcal{L}(\phi_i, \partial_\mu \phi_i)}{\partial (\partial_\mu \phi_i)} - (d_{\mu\nu} - \delta_{\mu\nu}) \left(x^{(-1)} \right)^\nu \frac{\partial \mathcal{L}(\phi_i, \partial_\mu \phi_i)}{\partial (\partial_\mu \phi_i)} = 0. \quad (7)$$

Here, $i = 1, 2, \dots, n$ is the number of degrees of freedom (*i.e.* scalar fields), the index μ runs from 1 to 4, $x^\mu = (x^1, x^2, x^3, x^4 = t)$ and $\partial_\mu \phi_i = \partial \phi_i / \partial x^\mu$. d is a diagonal matrix, the elements of which are the time and spatial dimensions $d = \text{diag}(1, d_{11}, d_{22} \dots d_{nn})$ with $D = \text{Tr}(d) - 1$ and δ is the diagonal unit matrix. The third term of the left hand side of (7) is the additional terme due to the fractional dimension. In an Euclidean space where $d_{\mu\nu} = \delta_{\mu\nu}$ this term vanishes.

4 Lagrangian formulation of waves propagation in porous media

Several models have been worked out to describe the acoustical wave propagation in fluid saturated porous media. Among them, the Biot's theory [4] is the most popular one. It takes into account the solid and fluid motions and their couplings through different processes. In [5], Johnson has established the most general Lagrangian density of the acoustical field in a porous medium leading to linear wave equations. In his formulation, the interacting fields are the fluid (U^f) and solid (U^s) displacements $U^{f,s}(\mathbf{r}, t)$ averaged within a representative elementary volume $V \sim l^3$ such that l is small compared to the relevant wavelength λ but large compared to the typical size of the pores a : $a \ll l \ll \lambda$. To derive linear equations of motion (*i.e.* wave equations), the Lagrangian density \mathcal{L} must contain terms up to the first order in fields derivatives. Furthermore, for homogeneous and isotropic systems, \mathcal{L} is a scalar quantity with no dependence on position and time. The only contributions to the Lagrangian density are the scalars defined from the space and time derivatives of displacements $U^{f,s}(\mathbf{r}, t)$. So, the most general Lagrangian density for homogeneous and isotropic systems involving two components has the following form:

$$\begin{aligned} \mathcal{L} = \frac{1}{2} \left\{ \rho_{11} \dot{\mathbf{U}}^s{}^2 + 2\rho_{12} (\dot{\mathbf{U}}^s \cdot \dot{\mathbf{U}}^f) + \rho_{22} \dot{\mathbf{U}}^f{}^2 \right. \\ \left. - \left[\alpha_1 (\vec{\nabla} \cdot \mathbf{U}^s)^2 + 2\alpha_2 (\vec{\nabla} \cdot \mathbf{U}^s) (\vec{\nabla} \cdot \mathbf{U}^f) \right. \right. \\ \left. \left. + \alpha_3 (\vec{\nabla} \cdot \mathbf{U}^f)^2 + \alpha_4 (\vec{\nabla} \times \mathbf{U}^s)^2 \right. \right. \\ \left. \left. + 2\alpha_5 (\vec{\nabla} \times \mathbf{U}^s) (\vec{\nabla} \times \mathbf{U}^f) + \alpha_6 (\vec{\nabla} \times \mathbf{U}^f)^2 \right] \right\}. \quad (8) \end{aligned}$$

In this equation, the kinetic energy is the sum of terms which contain times derivatives, while the remaining terms form the potential energy. The sets ρ_{ij} and α_i are the phenomenological parameters to be determined; they are related respectively to the densities of fluid and solid and to their elastic modulus.

5 Biot wave equations in a fractional dimensional space

Since the fluid filling the pores space of the porous medium does not experience any shear restoring force, nor does it contribute to one on the solid, we postulate that $\alpha_5 = \alpha_6 = 0$. In this case, when applied to the Lagrangian density (8), the generalized Euler-Lagrange equations (7) give the following coupled wave equations :

$$\begin{aligned} \rho_{11} \ddot{\mathbf{U}}^s + \rho_{12} \ddot{\mathbf{U}}^f = \alpha_1 \vec{\nabla} (\vec{\nabla} \cdot \mathbf{U}^s) + \alpha_2 \vec{\nabla} (\vec{\nabla} \cdot \mathbf{U}^f) \\ - \alpha_4 \vec{\nabla} \times (\vec{\nabla} \times \mathbf{U}^s) + \alpha_1 \Delta (\vec{\nabla} \cdot \mathbf{U}^s) \\ + \alpha_2 \Delta (\vec{\nabla} \cdot \mathbf{U}^f) - \alpha_4 \Delta \times (\vec{\nabla} \times \mathbf{U}^s) \quad (9) \end{aligned}$$

and

$$\begin{aligned} \rho_{22} \ddot{\mathbf{U}}^f + \rho_{12} \ddot{\mathbf{U}}^s = \alpha_3 \vec{\nabla} (\vec{\nabla} \cdot \mathbf{U}^f) + \alpha_2 \vec{\nabla} (\vec{\nabla} \cdot \mathbf{U}^s) \\ + \alpha_2 \Delta (\vec{\nabla} \cdot \mathbf{U}^s) + \alpha_3 \Delta (\vec{\nabla} \cdot \mathbf{U}^f). \quad (10) \end{aligned}$$

where

$$\Delta = \begin{pmatrix} \frac{d_{11}-1}{x} \\ \frac{d_{22}-1}{x} \\ \frac{d_{33}-1}{x} \\ x \end{pmatrix} \quad (11)$$

is the term of fractional dimensions.

In the following we study the propagation in a 2-coordinates anisotropic porous medium saturated by a fluid. The fractional dimension d_{11} is associated with the coordinate x , while the dimension attached to y coordinate is $d_{22} = 1$. Roughly speaking, the medium looks like those presented on Fig 1. Then, the vector of noninteger dimensions becomes

$$\Delta = \begin{pmatrix} \frac{d_{11}-1}{x} \\ 0 \\ 0 \end{pmatrix} \quad (12)$$

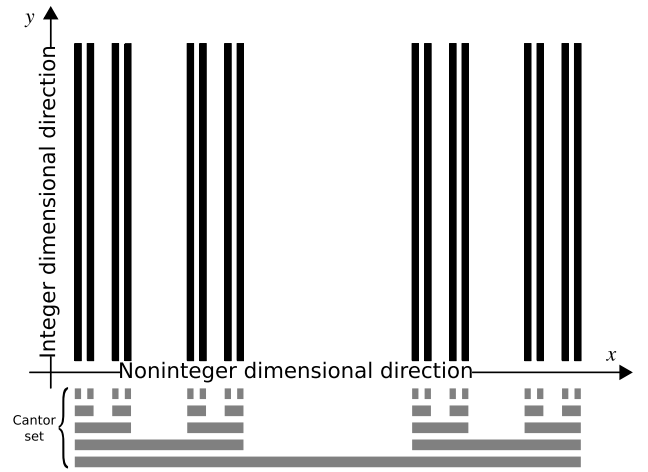


Figure 1: Shema of noninteger dimensional direction along ox axis

5.1 Longitudinal waves

As in the case of elastic solid, the dilatational waves are obtained by using scalar displacements potentials. Such waves disturb the medium by producing actual motion along the direction in which they travel. So, we consider acoustical waves which propagate along the ox axis, *i.e.* such that $\mathbf{U}^s = u_x^s \mathbf{e}_x$ and $\mathbf{U}^f = u_x^f \mathbf{e}_x$. In this case, (9) and (10) become

$$\begin{aligned} \rho_{11} \ddot{u}_x^s + \rho_{12} \ddot{u}_x^f = \alpha_1 \partial_{xx} u_x^s + \alpha_2 \partial_{xx} u_x^f \\ + \frac{d_{11}-1}{x} (\alpha_1 \partial_x u_x^s + \alpha_2 \partial_x u_x^f) \quad (13) \end{aligned}$$

and

$$\begin{aligned} \rho_{12} \ddot{u}_x^s + \rho_{22} \ddot{u}_x^f = \alpha_3 \partial_{xx} u_x^f + \alpha_2 \partial_{xx} u_x^s \\ + \frac{d_{11}-1}{x} (\alpha_2 \partial_x u_x^s + \alpha_3 \partial_x u_x^f). \quad (14) \end{aligned}$$

For a monochromatic time dependence, $u^i(x, t) = u_x^i(x) \cdot e^{j\omega t}$, these equations can be written in matrix notation as :

$$\omega^2 M \Phi + \mathbb{I} \partial_{xx} \Phi + \frac{d_{11}-1}{x} \mathbb{I} \partial_x \Phi = 0 \quad (15)$$

where $\Phi = \begin{pmatrix} u_x^s \\ u_x^f \end{pmatrix}$, \mathbb{I} is a diagonal unit matrix and M is given by

$$M = \begin{pmatrix} \alpha_1 & \alpha_2 \\ \alpha_2 & \alpha_3 \end{pmatrix}^{-1} \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{12} & \rho_{22} \end{pmatrix}. \quad (16)$$

The eigenvalues and eigenvectors of M give the longitudinal modes of propagation corresponding respectively to the fast (index F) and slow (index S) waves. For that we seek solution as $\Phi = x^a \Psi$. Writting $P^{-1}\Psi = \Upsilon = \begin{pmatrix} v_F \\ v_S \end{pmatrix}$, we get two uncoupled equations which are respectively a fast compression and a slow compression wave equations.

$$\partial_{xx} v_F + \left(\frac{1}{x}\right) \partial_x v_F + \left(\frac{-\Lambda^2}{x^2} + \omega^2 D_+\right) v_F = 0 \quad (17)$$

$$\partial_{xx} v_S + \left(\frac{1}{x}\right) \partial_x v_S + \left(\frac{-\Lambda^2}{x^2} + \omega^2 D_-\right) v_S = 0 \quad (18)$$

with $\Lambda = \frac{1}{2}(d_{11} - 2)$.

With the new variables, $z = \omega\sqrt{D_{\pm}}x$ where D_{\pm} are the eigenvalues do the matrix M , these equations become Bessel equations :

$$\partial_{zz} v_{F,S} + \left(\frac{1}{z}\right) \partial_z v_{F,S} + \left(1 - \frac{\Lambda^2}{z^2}\right) v_{F,S} = 0 \quad (19)$$

So, the solutions of equations 19 are Bessel's functions, $v_F(x) = J_{\Lambda}^F(x)$ and $v_S(x) = J_{\Lambda}^S(x)$. For fractional values of d_{xx} , Λ is non integer and the general solution of (17) and (18) are :

$$u_x^s(x, t) = \left(x^{1-\frac{d_{11}}{2}}\right) \left\{ (\lambda_F J_{\Lambda}^F(x) + \mu_F J_{-\Lambda}^F(x)) - (\lambda_S J_{\Lambda}^S(x) + \mu_S J_{-\Lambda}^S(x)) \right\} e^{j\omega t} \quad (20)$$

and

$$u_x^f(x, t) = \left(x^{1-\frac{d_{11}}{2}}\right) \left\{ p_{21} (\lambda_S J_{\Lambda}^S(x) + \mu_S J_{-\Lambda}^S(x)) - p_{22} (\lambda_F J_{\Lambda}^F(x) + \mu_F J_{-\Lambda}^F(x)) \right\} e^{j\omega t} \quad (21)$$

where p_{21} and p_{22} are related to the eigenvectors of the matrix M , and $\lambda_{F,S}$ and $\mu_{F,S}$ are constants to be determined from initial conditions.

5.2 Shear waves

The components of displacements produced by shear waves are $\mathbf{U}^s = u_y^s \mathbf{e}_y$ and $\mathbf{U}^f = u_y^f \mathbf{e}_y$. Equations of motion are then :

$$\rho_{11} \ddot{u}_y^s + \rho_{12} \ddot{u}_y^f = \alpha_4 \partial_{xx} u_y^s + \frac{d_{11}-1}{x} (\alpha_4 \partial_x u_y^s) \quad (22)$$

$$\rho_{12} \ddot{u}_y^s + \rho_{22} \ddot{u}_y^f = 0 \quad (23)$$

For a monochromatic time dependence, $u^i(x, t) = u_y^i(x) e^{j\omega t}$, (23) is a simple relation between the amplitudes of motions of the two phases of the porous medium :

$$u_y^f = -\frac{\rho_{12}}{\rho_{22}} u_y^s. \quad (24)$$

The meaning of this relation is that, because of the tortuosity of the porous medium, the fluid is dragged by the solid during its motion.

Substitution of 24 in equation 21, leads to:

$$\partial_{xx} u_y^s + \frac{d_{11}-1}{x} (\partial_x u_y^s) + \frac{\omega^2}{\alpha_4} \left(\rho_{11} - \frac{\rho_{12}^2}{\rho_{22}} \right) u_y^s = 0. \quad (25)$$

If we put $u_y^s(x) = x^a u_y^s$ the change of variable $z = \omega x \sqrt{\frac{1}{\alpha_4} \left(\rho_{11} - \frac{\rho_{12}^2}{\rho_{22}} \right)}$, leads to the Bessel's equation :

$$\partial_{zz} u_y^s + \left(\frac{1}{z}\right) \partial_z u_y^s + \left(1 - \frac{\Lambda^2}{z^2}\right) u_y^s = 0 \quad (26)$$

where $\Lambda = \frac{1}{2}(d_{11} - 2)$.

So, the solutions of equations (26) are Bessel's functions, $u_y^s = J_{\Lambda}(x)$. But, Λ is noninteger, so $J_{-\Lambda}(x)$ is also a solution.

So, the motion of solid and fluid produced by the shear wave are

$$u_y^s(x, t) = \left(x^{1-\frac{d_{11}}{2}}\right) (\lambda J_{\Lambda}(x) + \mu J_{-\Lambda}(x)) e^{j\omega t} \quad (27)$$

$$u_y^f(x, t) = -\frac{\rho_{12}}{\rho_{22}} \left(x^{1-\frac{d_{11}}{2}}\right) (\lambda J_{\Lambda}(x) + \mu J_{-\Lambda}(x)) e^{j\omega t} \quad (28)$$

where λ and μ are constants to be determined from the initial conditions.

6 Conclusion

We have derived the Biot wave equations in a fractional dimensional porous medium from the Stillinger-Palmer-Stavrinou formalism. When the propagation is along a non-integer dimensional coordinate, the amplitude variations are described by Bessel functions. This behaviour must be compared to the one of fractal sets which are governed by the scale invariance and power laws. Since this formalism allows to distribute freely the fractional dimensions between the coordinates, it is a way to investigate the wave propagation in porous media with anisotropic fractional dimensions in which new coupling processes can be induced.

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