Implementation of the high order asymptotic models of linings and coatings to the SAW spectra evaluation

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High order asymptotic models of the relatively thin linings and coatings are used to reduce the problem dimension when dealing with a reasonable frequency range. The essential point consists in the use of asymptotic expansion for the internal stress and strain state of a laminate, subjected to the tight conditions at least on one of its faces, i.e., the conditions formulated for the displacements. The respective algorithm, based on it, is shorter than for the direct evaluation of the SAW spectra. It can be implemented to the calculation of Rayleigh waves or Lamb waves in layered media. Numerical examples are presented with both asymptotical and numerical evaluation of error.

1 Introduction

Many researchers in NDE, seismology, electronic devices and other applications investigate the ultrasonic wave propagation in layered solids. Despite the existence of the general solution to this problem (e.g., see [1]) for the variety of applications dealing with the limited frequency band the consuming full analysis is not required. When the layer half-thickness h is much smaller than the minimal scale of process L (wavelength), i.e. $\varepsilon = h/L << 1$, the approximate model may accelerate the calculations with the satisfactory accuracy. In particular, the influence of lining or coating can be replaced by introducing the *impedance boundary conditions* (IBC) on the interface. Such low order IBC for anisotropic layer in between two solids are constructed in [2,3] by expanding the transfer matrix in power series. Below the alternative way to deduce IBC of high order is suggested. The displacements and stresses are sought in the form of asymptotic ε-power series with successive asymptotic integration of 3D dynamic boundary value problem of viscoelasticity. This method is well developed in the theory of thin plates and shells and provides an efficient way to deduce the relations of any order. A simplified version of a similar method with expansion in power series of transversal coordinate is applied in [4] for 2D problem of isotropic piezoelectric coating with a stress-free surface within the third order asymptotic accuracy. Another method to approximate waves propagating as $e^{i(kX - \omega T)}$ in layered solids is based on the recursive relations to deduce transfer matrices or stiffness matrices using Padé approximation together with Magnus expansion [5,6]. Our results differ from those in [5,6] by the high order terms [7] and provide the uniform asymptotic accuracy. We also apply the suggested method in the case of dissipative materials including non traditional nematic elastomers [8].

2 Formulation and scaling

Consider a laminate where each j th anisotropic ply occupies a region $-\infty < X_1, X_2 < \infty$, $Z_j \le Z \le Z_{j+1}$ $\left(H_j = Z_{j+1} - Z_j, j = 1, 2, \ldots, N\right)$. Let us denote its mass density by ρ_j and stiffness matrix by $\mathbf{G}_j = \left\| g_{pg}^j \right\|$ $\left(p, q = 1,2,3,4,5,6 \sim 11,22,33,23,13,12\right)$. We assume that the longitudinal displacements $\mathbf{U}_j \equiv \left(U_{1j}, U_{2j}\right)^T$, normal deflection $W_j \equiv U_{3j}$ and stresses σ_{pqj} satisfy the 3D equations of dynamic elasticity and conditions of full contact on the interfaces. On the faces $Z^- = Z_1$, $Z^+ = Z_{N+1}$ $\left(Z^+ - Z^- = 2h\right)$ the tight boundary conditions are assumed, i.e., their formulation includes the explicit displacement term U_p^- or U_p^+. Using small parameter ε perform the normalization $\mathbf{x} = \mathbf{X}/L$ $\left(\mathbf{X} = (X_1, X_2)\right)$, $z = Z/h$ and $t = T/T_0$ $\left(T_0 = \varepsilon^{\tau-1} L/c_0, c_0 \equiv \sqrt{E_0/\rho_0}\right)$. Here ρ_0 and E_0 are the reference mass density and stiffness, and τ is a parameter. Introduce the dimensionless quantities and operators

$$\mathbf{d}_j = \begin{bmatrix} w \\ u_2 \\ u_1 \end{bmatrix}_j, \quad \mathbf{t}_{zj} = \begin{bmatrix} \sigma_{zz} \\ \sigma_{2z} \\ \sigma_{1z} \end{bmatrix}_j, \quad \mathbf{t}_{xj} = \begin{bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{22} \end{bmatrix}_j,$$

$$\mathbf{d}^- = \mathbf{d}_1\big|_{z=z^-}, \quad \mathbf{d}^+ = \mathbf{d}_N\big|_{z=z^+}, \quad \mathbf{t}_z^- = \mathbf{t}_{z1}\big|_{z=z^-},$$

$$\mathbf{t}_z^+ = \mathbf{t}_{zN}\big|_{z=z^+}, \quad \mathbf{d}_{\pm} = \tfrac{1}{2}\left(\mathbf{d}^+ \pm \mathbf{d}^-\right), \quad \mathbf{t}_{\pm} = \tfrac{1}{2}\left(\mathbf{t}_z^+ \pm \mathbf{t}_z^-\right)$$

$$z_j^- = z_j, \quad z_j^+ = z_{j+1}, \quad z_j^0 = \tfrac{1}{2}\left(z_j^+ + z_j^-\right),$$

$$z^{\mp} \equiv z_1, z_{N+1}, \quad \sum_- \equiv \sum_{k=1}^{j-1}, \quad \sum_+ \equiv \sum_{k=j+1}^{N},$$

$$\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 & \partial_2 & \partial_1 \\ 0 & \partial_1 & \partial_2 & 0 & 0 \\ \partial_1 & \partial_2 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{G}_{0j} = \mathbf{G}\begin{pmatrix} 345 \\ 345 \end{pmatrix}_j,$$

$$\mathbf{G}_{*j} = \mathbf{G}\begin{pmatrix} 162 \\ 345 \end{pmatrix}_j, \quad \mathbf{G}_{\perp j} = \mathbf{G}\begin{pmatrix} 16245 \\ 162 \end{pmatrix}_j, \quad \mathbf{G}_{\|j} = \mathbf{G}\begin{pmatrix} 16245 \\ 345 \end{pmatrix}_j,$$

$$\mathbf{D}_{1j} = \mathbf{D}\mathbf{G}_{\|j} + \mathbf{G}_{\|j}^T \mathbf{D}^T, \quad \mathbf{D}_{2j} = \mathbf{D}\,\mathbf{G}\begin{pmatrix} 16245 \\ 16245 \end{pmatrix}_j \mathbf{D}^T,$$

$$\mathbf{N}_j = \mathbf{D}\mathbf{G}_{\|j}\mathbf{G}_{0j}^{-1}, \quad \mathbf{A}_j = \frac{\rho_j}{\rho_0}\partial_t^2 - \mathbf{D}_{2j}, \quad \mathbf{B}_j = \mathbf{A}_j + \mathbf{D}_{1j}\mathbf{G}_{0j}^{-1}\mathbf{D}_{1j}.$$

The desired displacements and stresses for the internal stress and strain state is sought in the form of asymptotic series

$$\mathbf{U}_j = h\varepsilon^\lambda \left\{\mathbf{u}^0 + \varepsilon\mathbf{u}^1 + \ldots\right\}_j, \quad W_j = h\varepsilon^\lambda \left\{w^0 + \varepsilon w^1 + \ldots\right\}_j, \quad (1)$$

$$\sigma_{pqj} = E_0\varepsilon^\lambda \left\{\sigma_{pq}^0 + \varepsilon\sigma_{pq}^1 + \ldots\right\}_j, \quad (2)$$

with undetermined power λ. The Hooke law, elasticity equations and interface conditions yield

$$\varepsilon_{\alpha\alpha j}^s = \partial_\alpha u_{\alpha j}^{s-1} \ (\alpha, \beta = 1,2), \quad \varepsilon_{zzj}^s = \partial_z w_j^s,$$

$$\gamma_{12j}^s = \partial_1 u_{2j}^{s-1} + \partial_2 u_{1j}^{s-1}, \quad \gamma_{\alpha zj}^s = \partial_\alpha w_j^{s-1} + \partial_z u_{\alpha j}^s,$$

$$\mathbf{t}_{zj}^s = \mathbf{G}_{\|j}^T \mathbf{D}^T \mathbf{d}_j^{s-1} + \mathbf{G}_{0j}\partial_z \mathbf{d}_j^s, \quad \mathbf{t}_{xj}^s = \mathbf{G}_{\perp j}^T \mathbf{D}^T \mathbf{d}_j^{s-1} + \mathbf{G}_{*j}\partial_z \mathbf{d}_j^s,$$

$$\partial_z^2 \mathbf{d}_j^s = \mathbf{G}_{0j}^{-1}\left(-\mathbf{D}_{1j}\partial_z \mathbf{d}_j^{s-1} + \mathbf{A}_j \mathbf{d}_j^{s-2}\right),$$

$$\mathbf{t}_{zj}^s = \mathbf{t}_{zj+1}^s, \quad \mathbf{d}_j^s = \mathbf{d}_{j+1}^s \quad \left(z = z_{j+1}; j = 1,2,....,N-1\right),$$

$$\mathbf{t}_z^{s,\mp} = \mathbf{t}_z^{\mp}\delta_0^{\lambda+s}, \quad \mathbf{d}^{s,\mp} = \mathbf{d}^{\mp}\delta_0^{\lambda+s+1} \quad \left(z = z^{\mp}\right).$$

(δ_β^α is a Kronecker delta). It is easily to check that this chain of relations looses recurrence at $\tau > 1$, so we choose the maximal value $\tau = 1$ with T_0 corresponding to the minimal possible period in plies. The value λ follows for the order of the facial load. Upon the recurrent relations we successively determine quantities for $s = 0,1,\dots$ by setting

$$\mathbf{d}_j^k = \sum_{m=0}^{k+1}\frac{z^m}{m!}\mathbf{d}_{mj}^k, \quad \mathbf{t}_{zj}^1 = \sum_{l=0}^{k}\frac{z^m}{m!}\mathbf{t}_{mzj}^k. \tag{3}$$

3 Asymptotic integration

On performing the substitution of series (3) into recurrent chain, expressing $\mathbf{d}_\pm^k$ and $\mathbf{t}_\pm^k$ on the faces using equations of motion and conditions on the interfaces we obtain for the sum of three terms in (1), (2)

$$\mathbf{d}_- = \left(\mathbf{G}_0^{-1} + \mathbf{C} + \mathbf{T}\right)\mathbf{t}_+ - \left(0 - \mathbf{N} + \mathbf{E}\right)^T\mathbf{d}_+, \tag{4}$$

$$\mathbf{t}_- = \left(0 - \mathbf{N} - \mathbf{F}\right)\mathbf{t}_+ + \left(0 + 0 + \mathbf{L}\right)\mathbf{d}_+, \tag{5}$$

$$\mathbf{t}_- + \left[0 + \mathbf{N} + \left(\mathbf{F} \pm \mathbf{L}_2\right)\right]\mathbf{t}_+ = \left(0 + 0 + \mathbf{L}\right)\mathbf{d}^\pm, \tag{6}$$

$$\mathbf{d}_- + \left[0 + \mathbf{N} + \left(\mathbf{E} \pm \mathbf{L}_2\right)\right]^T\mathbf{d}_+ =$$
$$= \left[\mathbf{G}_0^{-1} + \left(\mathbf{C} \pm \mathbf{G}_0^{-1}\mathbf{N}\right) + \left(\mathbf{T} + \mathbf{G}_0^{-1}\mathbf{N}^2 \pm \mathbf{G}_0^{-1}\mathbf{F} \pm \mathbf{C}\mathbf{N}\right)\right]\mathbf{t}_z^\pm, \tag{7}$$

where

$$\mathbf{G}_0^{-1} \equiv \frac{1}{2}\sum_j h_j\mathbf{G}_{0j}^{-1}, \quad \mathbf{G}_{\pm j}^{-1} \equiv \frac{1}{2}\sum_\pm h_k\mathbf{G}_{0k}^{-1}, \quad \mathbf{N} = \frac{1}{2}\sum_j h_j\mathbf{N}_j,$$

$$\mathbf{N}_j^\pm \equiv \frac{1}{2}\sum_\pm h_k\mathbf{N}_k, \quad \mathbf{C}_j = \mathbf{N}_j^T\left(\mathbf{G}_{+j}^{-1} - \mathbf{G}_{-j}^{-1}\right) + \mathbf{G}_{0j}^{-1}\left(\mathbf{N}_j^+ - \mathbf{N}_j^-\right),$$

$$\mathbf{C} = \frac{1}{2}\sum_j h_j\mathbf{C}_j, \quad \mathbf{C}_j^\pm = \frac{1}{2}\sum_\pm h_k\mathbf{C}_k, \quad \mathbf{L}_j = \mathbf{A}_j + \mathbf{N}_j\mathbf{G}_{0j}\mathbf{N}_j^T,$$

$$\mathbf{L} = \frac{1}{2}\sum_j h_j\mathbf{L}_j, \quad \mathbf{L}_j^\pm = \frac{1}{2}\sum_\pm h_k\mathbf{L}_k, \quad \mathbf{L}_2 = \mathbf{L}\mathbf{G}_0^{-1},$$

$$\mathbf{F}_j = \mathbf{N}_j\left(\mathbf{N}_j^+ - \mathbf{N}_j^-\right) + \mathbf{L}_j\left(\mathbf{G}_{+j}^{-1} - \mathbf{G}_{-j}^{-1}\right), \quad \mathbf{F} = \frac{1}{2}\sum_j h_j\mathbf{F}_j,$$

$$\mathbf{F}_j^\pm = \frac{1}{2}\sum_\pm h_k\mathbf{F}_k, \quad \mathbf{E}_j^T = \mathbf{G}_{0j}^{-1}\left\{\mathbf{L}_j^+ - \mathbf{L}_j^-\right\} + \mathbf{N}_j^T\left\{\mathbf{N}_{+j}^T - \mathbf{N}_{-j}^T\right\},$$

$$\mathbf{E} = \frac{1}{2}\sum_j h_j\mathbf{E}_j, \quad \mathbf{E}_j^\pm = \frac{1}{2}\sum_\pm h_k\mathbf{E}_k.$$

The relations (4)-(7) remain the same in the dimensional form. The asymptotic error of these relations is determined by the order of the neglected terms and equals $O\left(\varepsilon^3\right)$. For the case of equivalent plies it is easily to increase the accuracy up to $O\left(\varepsilon^6\right)$. The respective relations acquire the following dimensional form

$$\mathbf{t}_- = \left(-\mathbf{N}_j\mathbf{G}_{0j} + h^2/3\,\mathbf{A}_{3j} - h^4/5\,\mathbf{A}_{5j}\right)\mathbf{d}_- +$$
$$+ \left(h\mathbf{A}_j - h^3/3\,\mathbf{A}_{4j}\right)\mathbf{d}_+, \tag{8}$$

$$h\mathbf{t}_+ = \left(\mathbf{G}_{0j} + h^2/3\,\mathbf{B}_j - h^4/5\,\mathbf{B}_{4j}\right)\mathbf{d}_- +$$
$$+ \left(h\mathbf{G}_{0j}\mathbf{N}_j^T - h^3/3\,\mathbf{B}_{3j} + h^5/6\,\mathbf{B}_{5j}\right)\mathbf{d}_+, \tag{9}$$

$$\mathbf{t}_- + h\left(\mathbf{N}_j \pm h\mathbf{L}_{2j} + h^2\mathbf{S}_{3j} \pm h^3\mathbf{R}_{4j} + h^4\mathbf{P}_{5j}\right)\mathbf{t}_+ =$$
$$= h\left(0 + \mathbf{L}_j \pm h\mathbf{L}_j\mathbf{N}_j^T + h^2\mathbf{M}_{4j} \pm h^3\mathbf{Q}_{5j}\right)\mathbf{d}^\pm, \tag{10}$$

$$\mathbf{d}_- + h\left(\mathbf{N}_j \pm h\mathbf{L}_{2j} + h^2\mathbf{S}_{3j} \pm h^3\mathbf{R}_{4j} + h^4\mathbf{P}_{5j}\right)^T\mathbf{d}_+ =$$
$$= h\mathbf{G}_{0j}^{-1}\left(1 \pm h\mathbf{N}_j + h^2\mathbf{N}_{2j} \pm h^3\mathbf{R}_{3j} + h^4\mathbf{T}_{4j} \pm h^5\mathbf{Y}_{5j}\right)\mathbf{t}_z^\pm, \tag{11}$$

where

$$\mathbf{A}_{3j} \equiv \mathbf{A}_j\mathbf{G}_{0j}^{-1}\mathbf{D}_{1j}, \quad \mathbf{B}_{3j} \equiv \mathbf{A}_{3j}^T,$$

$$\mathbf{S}_{3j} \equiv \left[\mathbf{L}_j\mathbf{N}_j^T - \left(\mathbf{A}_{3j} + \mathbf{N}_j\mathbf{B}_j\right)/3\right]\mathbf{G}_{0j}^{-1}, \quad \mathbf{A}_{4j} \equiv \mathbf{A}_j\mathbf{G}_{0j}^{-1}\mathbf{A}_j,$$

$$\mathbf{B}_{4j} \equiv 2/3\,\mathbf{B}_{3j}\mathbf{G}_{0j}^{-1}\mathbf{D}_{1j} + 1/4\,\mathbf{B}_j\mathbf{G}_{0j}^{-1}\mathbf{B}_j,$$

$$\mathbf{M}_{4j} \equiv \mathbf{S}_{3j}\mathbf{G}_{0j}\mathbf{N}_j^T - \left(\mathbf{A}_{4j} + \mathbf{N}_j\mathbf{B}_{3j}\right)/3,$$

$$\mathbf{R}_{4j} \equiv \left(\mathbf{M}_{4j} - \mathbf{L}_{2j}\mathbf{B}_j/3\right)\mathbf{G}_{0j}^{-1},$$

$$\mathbf{A}_{5j} \equiv 2/3\,\mathbf{A}_{4j}\mathbf{G}_{0j}^{-1}\mathbf{D}_{1j} + 1/4\,\mathbf{A}_{3j}\mathbf{G}_{0j}^{-1}\mathbf{B}_j,$$

$$\mathbf{B}_{5j} \equiv 4/5\,\mathbf{B}_{3j}\mathbf{G}_{0j}^{-1}\mathbf{A}_j + 2/15\,\mathbf{B}_j\mathbf{G}_{0j}^{-1}\mathbf{B}_{3j},$$

$$\mathbf{Q}_{5j} \equiv \mathbf{R}_{4j}\mathbf{G}_{0j}\mathbf{N}_j^T - \mathbf{L}_{2j}\mathbf{B}_{3j}/3,$$

$$\mathbf{P}_{5j} \equiv \left[\mathbf{Q}_{5j} - \mathbf{S}_{3j}\mathbf{B}_j/3 + \left(\mathbf{A}_{5j} + \mathbf{N}_j\mathbf{B}_{4j}\right)/5\right]\mathbf{G}_{0j}^{-1},$$

$$\mathbf{R}_{3j} \equiv \mathbf{S}_{3j} + \mathbf{N}_{2j}\mathbf{N}_j - \mathbf{L}_j\mathbf{N}_j^T\mathbf{G}_{0j}^{-1},$$

$$\mathbf{T}_{4j} \equiv \mathbf{R}_{3j}\mathbf{N}_j + \mathbf{N}_j\mathbf{R}_{3j} - \mathbf{N}_j\mathbf{N}_{2j}\mathbf{N}_j +$$
$$+ \left(\mathbf{B}_{4j}/5 + \mathbf{B}_j\mathbf{G}_{0j}^{-1}\mathbf{B}_j/9\right)\mathbf{G}_{0j}^{-1},$$

$$\mathbf{Y}_{5j} \equiv \mathbf{T}_{4j}\mathbf{N}_j +$$
$$+ \left[\left(\mathbf{A}_{5j} + \mathbf{N}_j\mathbf{B}_{4j}\right)/5 - \left(\mathbf{N}_{2j}\mathbf{A}_{3j} + \mathbf{R}_{3j}\mathbf{B}_j\right)/3\right]\mathbf{G}_{0j}^{-1}.$$

4 Modeling lining and coating

Let us now proceed to the consideration of the contact between the thin laminate and thicker solid, e.g., such as half space, thick plate or very shallow shell. In a thick solid we do not assume the existence of any small parameter and the scale of process is determined only by L. Thus, the scaling for thick solid differs only by transversal coordinate $z = Z/L$ and does not change the type of equations of motions and elasticity relations. The stress and strain state can be represented in the form of asymptotic series

$$\mathbf{U}_j = L\varepsilon^\mu\left\{\mathbf{u}^0 + \varepsilon\mathbf{u}^1 + \dots\right\}_j, \quad W_j = L\varepsilon^\mu\left\{w^0 + \varepsilon w^1 + \dots\right\}_j, \tag{12}$$

$$\sigma_{pqj} = E_0\varepsilon^\mu\left\{\sigma_{pq}^0 + \varepsilon\sigma_{pq}^1 + \dots\right\}_j, \tag{13}$$

by virtue of contact with thin laminate and the relations for s th order terms are the same as in 3D elasticity without additional recurrence. By this reason we do not perturb the procedure of asymptotic integration from Sec.3 and obtain the same results (4)-(11), where $\mathbf{t}_z^\pm$ and $\mathbf{d}^\pm$ are values on the interfaces (or on the face). The asymptotic error also remains the same. The interpretation depends on the type of boundary value problem considered: the lining is described by Eq.(4),(5) (laminate) or (8),(9) (single layer); the coating with given facial stresses is described by Eq.(6) (or (10)) and the coating with given facial displacements is described by Eq.(7) (or (11)), respectively. Thus, the influence of a lining or coating is replaced by approximated IBC.

5   Dissipative materials

The results (4)-(11) are obtained for pure elasticity. Assume now that the energy dissipation is allowed. A simplest case is the Kelvin-Voight model of linear viscoelasticity

$$\sigma_{pj} = \left\{ g'^{j}_{pq} + g''^{j}_{pq} \partial_t \right\} \varepsilon_{qj}$$
$$(p,q = 1,2,3,4,5,6 \sim 11,22,33,23,13,12)$$

where $\left\| g'^{j}_{pq} \right\|$ is a stiffness tensor and $\left\| g''^{j}_{pq} \right\|$ is a viscosity tensor. The viscosity coefficients are supposed to be small and the relaxation time $\tau^{j}_{pq} = g''^{j}_{pq} / g'^{j}_{pq} \ll 1$, so for the time harmonic problem $e^{-i\omega t}$, $\partial_t \sim -i\omega$ up to frequency

$$\omega \tau^{j}_{pq} = 2\pi \tau^{j}_{pq} / T_0 = O(1),$$

$$g^{j}_{pg} = g'^{j}_{pq} \left(1 - i 2\pi \varepsilon c_0 \tau^{j}_{pq} / h \right) = O(1).$$

This estimate holds even for the order $\tau^{j}_{pq} \sim L/c_0$. The asymptotic consideration is similar to the previous one and it is sufficient to set $\mathbf{G}_j = \mathbf{G}_j(\omega)$ and $\sigma_{pj} = g^{j}_{pq}\varepsilon_{qj}$ in (4)-(11). The deduction logic remains the same for more complicated case of complex modulii

$$\mathbf{G}_j = \mathbf{G}^0_j + \varepsilon \mathbf{G}^1_j + \varepsilon^2 \mathbf{G}^2_j + \dots.$$

Thus, in (4)-(11) the nonlinear viscosity is also possible at least for small enough ε. As an example of non traditional material let us consider the lining and coating made of liquid crystalline *nematic elastomer* [6]. Under wide assumptions such material can be treated as compressible, transversely isotropic with main "director" axis denoted by x_3 and having additional degrees of freedom $\boldsymbol{\theta} = (\theta_1, \theta_2)^{\mathrm{T}}$, describing the relative rotation of director $\mathbf{n}$ and material point. The rotation angle satisfies the quasistatic equation

$$\mathbf{n} \times \left[D_1 (1 + \tau_1 \partial_t) \mathbf{n} \times \boldsymbol{\theta} + D_2 (1 + \tau_2 \partial_t) \mathbf{n} \cdot \boldsymbol{\varepsilon} \right] = 0,$$

where D_1, D_2 and τ_1, τ_2 are the rotational stiffnesses and relaxation times. For the time harmonic problem

$$\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \frac{D_2}{D_1} \frac{1 - i\omega\tau_2}{1 - i\omega\tau_1} \begin{bmatrix} -\varepsilon_{23} \\ \varepsilon_{13} \end{bmatrix}.$$

The stiffness tensor are expressed via elastic constants c_k $(k = 1, \dots, 5)$ of transversal isotropy which yields stresses

$$g'_{11} = g'_{22} = \frac{2}{9}c_1 - \frac{4}{3}c_2 + 2c_3 + \frac{20}{9}c_4,$$

$$g'_{12} = \frac{2}{9}c_1 - \frac{4}{3}c_2 + 2c_3 - \frac{16}{9}c_4,$$

$$g'_{13} = g'_{23} = -\frac{4}{9}c_1 + \frac{2}{3}c_2 + 2c_3 - \frac{4}{9}c_4,$$

$$g'_{33} = \frac{8}{9}c_1 + \frac{8}{3}c_2 + 2c_3 + \frac{8}{9}c_4, \ g'_{44} = g'_{55} = 2c_5, \ g'_{66} = 2c_4,$$

$$\sigma_{pp} = (1 - i\omega\tau_R)(g'_{p1}\varepsilon_{11} + g'_{p2}\varepsilon_{22} + g'_{p3}\varepsilon_{33}) \ (p = 1,2,3)$$

$$\sigma_{q3} = (1 - i\omega\tau_R)g^{R}_{44}\gamma_{23}, \ (q = 1,2), \ \sigma_{12} = (1 - i\omega\tau_R)g'_{66}\gamma_{12},$$

$$c^{R}_5(\omega) = \frac{1}{2}g^{R}_{44} = c_5 - \frac{D^2_2}{8D_1} \frac{(1 - i\omega\tau_2)^2}{(1 - i\omega\tau_1)(1 - i\omega\tau_R)},$$

with the bulk relaxation time τ_R satisfying inequality $D^2_2 \tau_2 \leq 8c_5 D_1 \tau_R \tau_1$. The ideal nematic with $c^{R}_5(\omega = 0) = 0$ is closer to liquid. As seen the procedure from above is applicable for nematics under $\omega \max(\tau_1, \tau_2, \tau_R) \ll 1$ and relations (4)-(11) remain in force.

6   Numerical validation

As the numerical tests it is reasonable to consider the simplest partial waves for which the exact solution is easily to obtain for comparison. The typical results for waves falling from the top isotropic half space "+" and propagating through the two layered lining $(H_1 = H_2)$ into bottom half space "-" are presented in Figs.1,4,7. The relative error e is introduced as follows

$$e^2 = 1/4 \sum_{[+,-;P,S]} \left| 1 - M^{as}/M^{ex} \right|^2,$$

where M^{ex} runs the exact complex magnitudes and M^{as} runs the approximates values of the wave magnitudes in half-spaces obtained by using IBC. For the top coated half-space (see Figs.2,3,5,6,8,9) the error is chosen as

$$e^2 = 1/2 \sum_{[+,-;P,S]} \left| 1 - M^{as}/M^{ex} \right|^2,$$

for magnitudes of the reflected waves.

The materials are: Al (aluminium), Po (polystyrene), Eg (orthotropic E-glass). The curve number corresponds to the asymptotic order of the truncation in IBC (4)-(7).
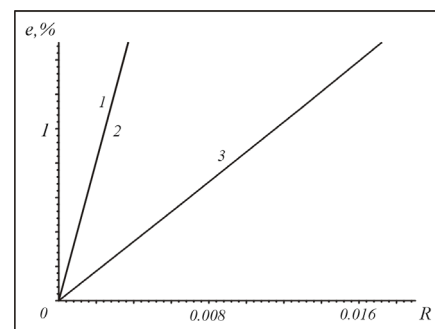

Fig.1 Error for S-wave falling at the angle 70° to the interface of Al-Eg0°/90°-Po against $R = H_1/L_1 + H_2/L_2$.
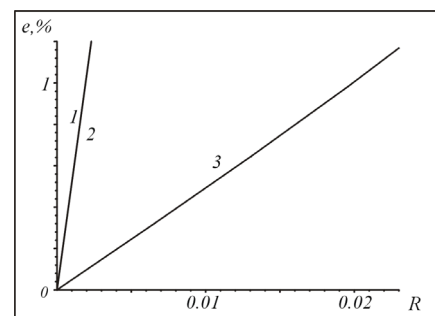

Fig.2 Error for S-wave falling at 70° to the interface of Al-Eg0°/90° (stress free coating face).

As seen the error is maximal for the free face, minimal for the clamed face and the interval of 1% model error is about 10^{-2}. This value is similar to the validity interval of the classical theory of plates. The accuracy is much better when the lining (or coating) becomes a single layer, for which the results are shown in Figs.4-6. As one can see the third order model has a validity interval 0.1 and the high order model is valid till 0.25, which reminds us the analogue with the high order classical plate theories. The improvement by third order is explained by the appearance of the wave operator in (4)-(11), since the less order models are quasistatic.
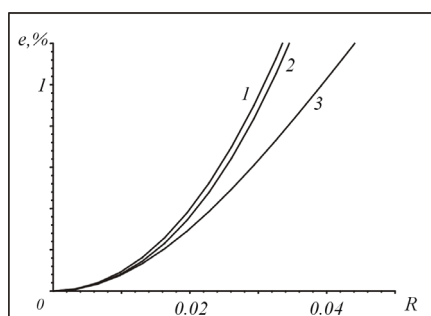
Fig.3 Error for S-wave falling at 70° to the interface of Al-Eg0^o/90° (clamped coating face).

Fig.4 Error for S-wave falling at 80° to the interface of Al-Eg0^o-Po (L is a min shear wavelength in E-glass).
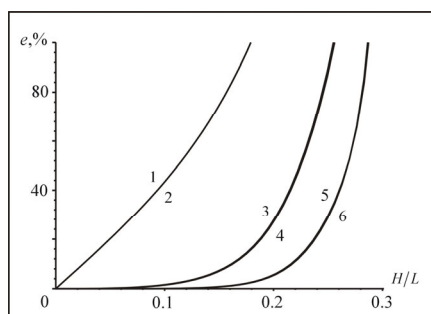
Fig.5 Error for S-wave falling at 80° to the interface of Al-Eg0^o (stress free coating face).

The tests for a single layer nematic coating and lining are shown in Figs.7-9. The parameters of non-ideal nematic are $D_1 = \mu_0 (r-1)^2 / r + A_0 \mu_0 Q$, $D_2 = \mu_0 (1-r^2)/r$ ($A_0 = 0.01$ is a deviation coefficient) with the anisotropy parameter of molecular chain $r \approx 3$ and $Q = 0.15 T^{0.3}$ under temperature $T = 37^{\circ}$C ; $\rho = 10^3$ [kg/m^3], $\tau_1 = 0.01$, $\tau_2 = 5 \times 10^{-5}$, $\tau_R = 10^{-6}$ [sec], $c_5 = \mu_0 (r+1)^2 / 8r$, $c_3 = 10^9$, $c_1 = 2c_2 = 2c_4 = \mu_0 = 10^5$ [N/m^2].

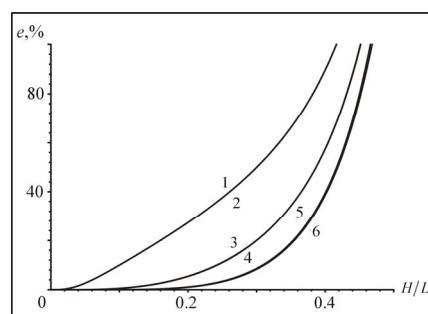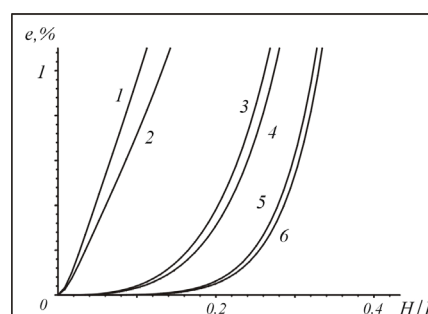Fig.6 Error for S-wave falling at 80° to the interface of Al-Eg0^o (clamped coating face).

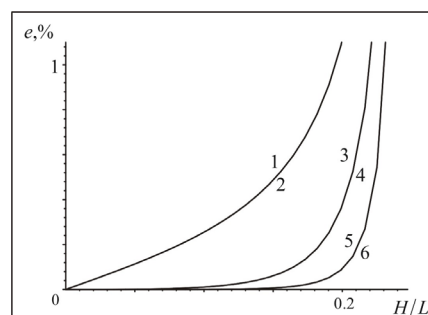Fig.7 Error for S-wave falling at the angle 70° to the interface of Al-Ne0^o-Po.

Fig.8 Error for S-wave falling at the angle 70° to the interface of Al-Ne0^o (stress free coating face).
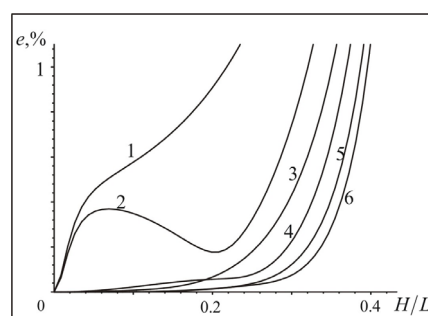

Fig.9 Error for S-wave falling at the angle 70° to the interface of Al-Ne0^o (clamped coating face).

The director orientation is chosen along z-axis but the error remains stable for other orientations. As one can see the validity interval is approximately the same as previously.

Since the surface and guided waves are composed of partial waves the expected accuracy should be of the same order. Note, that from the viewpoint of the thicker solid our model can be not low frequency. In Figs.10,11 the fundamental bending mode (B0) and first mode (B1) are presented for a three layered plate made of two aluminium layers and a lining of one layer of epoxy which is ten times thinner than each Al-ply. The curves are evaluated directly and using IBC (8),(9); the latter are marked by numbers as previously and c_S is the speed of shear wave in aluminium.
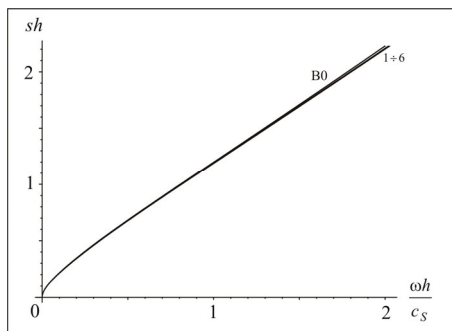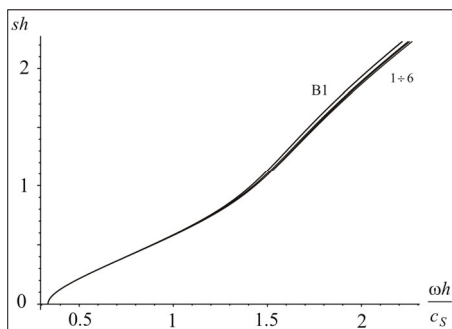

Fig.10 Fundamental bending mode.

Fig.11 First bending mode.

As shown the agreement is achieved not only for small frequency but for more than twice the total cut-off frequency. Of course, the most general situation depends on the limit behaviour of dispersion curves and on the energy flow between the layer and substrate (see, e.g., [9]).

7   Conclusion

The asymptotic high order theory of relatively thin lining and coating is suggested. The main result consists in replacing the lining (coating) by IBC which reduces the dimension of the respective boundary value problem. The physical reason of this fact is the absence of the fundamental mode in the spectrum of laminate under tight conditions on its faces. The obtained model is generally low frequency but possibly not long wave. It is applicable to the wide class of advanced composed materials, including those with energy dissipation and with non classical

behaviour. As shown, the nematic liquid crystalline elatomers can be approximated in such a way. Besides the consideration of the full interface contact one can similarly deduce the respective IBC for a slipping interface, frictional contact, etc.

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