Three dimensional orthogonality of the Lamb modes in layered plates of elastic and viscoelastic materials and their implementation to the far field evaluation

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The 3D guided waves in the linearly viscoelastic laminates are considered. On the plate surfaces any of the homogeneous boundary conditions are allowed, e.g., the Lamb waves, waves in clamped plates, etc. are taken into account. The fundamental property of these waves is their generalized orthogonality, which is deduced and discussed. The applications of the orthogonality relations for solving some particular boundary value problems are suggested. A method for the exact calculation of the far field caused by an acoustic source of a finite size is suggested. The only restriction is that the distance required must exceed the longitudinal radius of the source. The obtained results can be used for evaluating the fields radiated by ultrasonic transducers of arbitrary aperture and by other realistic sources.

1 Introduction

The wide use of composite materials causes the increasing attention to the guided waves in layered plates, which is a subject of monographs, reviews and numerous papers. As known, such guided waves are not orthogonal like trigonometrical Fourier series but they possess the orthogonality relations (OR) with respect to the power flow. These OR were deduced in the 70's for an elastic strip with various homogeneous boundary conditions on its faces [1-7]. The relations for 3D guided waves are presented below. Such OR can be used to construct the linear algebraic system of equations with respect to the unknown mode coefficients when using mode decomposition similarly to the various plane problems, e.g., the contact interaction between strips and a half-space, diffraction by a crack or wave reflection by an edge. In this paper the 3D guided waves are considered in a laminate with homogeneous boundary conditions on its faces (HBCF) including stress free faces, fixed faces or any other combinations of zero displacements or zero stresses providing the total energy reflection by the faces. The viscoelasticity is taken into account in the form of Kelvin-Voigt model or Maxwell model. The main motivation for this study is to generalise the results obtained earlier for one layer and pure elasticity [8-10], to elucidate the physics and to work out a method for exact calculation of the field, radiated by a realistic acoustic source into viscoelastic laminate. Since the numerical methods for 3D problems are time consuming the analytical and semi-analytical methods are still of interest for NDT needs when modelling the far-field and near-field.

2 Formulation

Consider a laminate composed of N plies where each j th ply occupies a region $-\infty < x_1, x_2 < \infty$, $z_j \le x_3 \le z_{j+1}$ and subjected to the time-harmonic load (Fig.1). To be brief the factor $e^{-i\omega t}$ is omitted in what follows. The layer displacements u^j_α satisfy the equations of motion

$$\partial_\beta \sigma^j_{\alpha\beta} + \rho_j \omega^2 u^j_\alpha + f^j_\alpha = 0\,, \quad (\alpha, \beta = 1,2,3) \tag{1}$$

where ρ_j are mass densities and f^j_α are body forces to be specified further. The stresses $\sigma^j_{\alpha\beta}$ and strains $\varepsilon^j_{\alpha\beta}$ satisfy Hook's law and Kelvin-Voigt model of viscoelasticity with the complex-valued Lame constants, wave numbers and wave speeds

$$\sigma^j_{\alpha\beta} = c'^j_{\alpha\beta\gamma\delta}\varepsilon^j_{\gamma\delta} + c''^j_{\beta\beta\gamma\delta}\dot{\varepsilon}^j_{\gamma\delta}\,, \quad c''^j_{\beta\beta\gamma\delta} << 1\,, \tag{2}$$

$$\varepsilon^j_{\alpha\beta} = 1/2\left\{\partial_\beta u^j_\alpha + \partial_\alpha u^j_\beta\right\}, \quad \dot{\varepsilon}^j_{\alpha\beta} = -i\omega\varepsilon^j_{\alpha\beta}\,. \tag{3}$$

$$\lambda_j = \lambda'_j - i\omega\lambda''_j\,, \quad \mu_j = \mu'_j - i\omega\mu''_j\,, \tag{4}$$

$$\left\{c^j_P\right\}^2 = \left(\lambda_j + 2\mu_j\right)\rho_j^{-1}, \left\{c^j_S\right\}^2 = \mu_j\rho_j^{-1}, k^j_{P,S} = \omega/c^j_{P,S}\,. \tag{5}$$

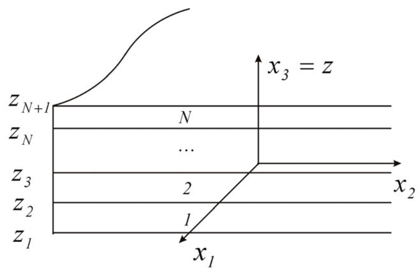

Fig.1 Laminate geometry.

On the interface $x_3 = z_j$, $j = 2,3,\ldots,N$ the conditions of the full contact are assumed

$$\sigma^{j-1}_{\alpha 3} = \sigma^j_{\alpha 3}\,, \quad u^{j-1}_\alpha = u^j_\alpha\,. \tag{6}$$

In addition the field may satisfy the conditions on the faces $z^- = z_1$ and $z^+ = z_{N+1}$ in the form of given stresses $\sigma^{\mp}_{\alpha 3}$ or displacements $u^{\mp}_\alpha$ or by their combinations.

3 Field representation

Introduce the displacement field and proceed to the cylindrical coordinates r, θ, z : $x_1 = r\cos\theta$, $x_2 = r\sin\theta$, $x_3 = z$. Using Lame potentials and separation of variables at the absence of body forces, the waves propagating in r-direction in j th layer result as follows

$$u^j_r = \left[-u^j B'_n + w^j \frac{n}{sr} B_n\right]\begin{Bmatrix}\cos n\theta \\ -\sin n\theta\end{Bmatrix}, \tag{7}$$

$$u^j_\theta = \left[u^j \frac{n}{sr} B_n - w^j B'_n\right]\begin{Bmatrix}\sin n\theta \\ \cos n\theta\end{Bmatrix}, \tag{8}$$

$$u^j_z = v^j B_n \begin{Bmatrix}\cos n\theta \\ -\sin n\theta\end{Bmatrix}. \tag{9}$$

The terms $B_n \equiv B_n(sr)$ are any of the appropriate Bessel function or Hankel function of the first or second kind and $B'_n \equiv dB_n/d\{sr\}$. The functions $u^j(z)$, $v^j(z)$ and $w^j(z)$ satisfy the system of equations

$$\left[\frac{d^2}{dz^2}+\alpha_j\left\{q_P^j\right\}^2\right]u^j-\gamma_j s\frac{dv^j}{dz}=0\,, \qquad (10)$$

$$\left[\alpha_j\frac{d^2}{dz^2}+\left\{q_S^j\right\}^2\right]v^j+\gamma_j s\frac{du^j}{dz}=0\,, \qquad (11)$$

$$\frac{d^2w^j}{dz^2}+\left\{q_S^j\right\}^2 w^j=0\,, \qquad (12)$$

$$\left\{q_S^j\right\}^2\equiv\left\{k_S^j\right\}^2-s^2\,,\quad \left\{q_P^j\right\}^2\equiv\left\{k_P^j\right\}^2-s^2\,. \qquad (13)$$

where $\alpha_j\equiv 2+\beta_j$, $\beta_j\equiv\lambda_j/\mu_j$, $\gamma_j\equiv\beta_j+1$. Equations (7)-(13) permit us to describe the guided wave of the wavenumber s within constant factor in a simple form $\left(A_{P,S}^{\pm j},B_S^{\pm j}=const\right)$

$$\begin{bmatrix}u^j\\ v^j\end{bmatrix}=A_P^{+j}\begin{bmatrix}\cos q_P^j z\\ \frac{q_P^j}{s}\sin q_P^j z\end{bmatrix}+A_P^{-j}\begin{bmatrix}\sin q_P^j z\\ -\frac{q_P^j}{s}\cos q_P^j z\end{bmatrix}+$$
$$+A_S^{+j}\begin{bmatrix}-\frac{q_S^j}{s}\cos q_S^j z\\ \sin q_S^j z\end{bmatrix}+A_S^{-j}\begin{bmatrix}\frac{q_S^j}{s}\sin q_S^j z\\ \cos q_S^j z\end{bmatrix}, \qquad (14)$$

$$w^j=B_S^{+j}\cos q_s^j z+B_S^{-j}\sin q_s^j z\,. \qquad (15)$$

The stresses look as follows (not to sum over j)

$$\frac{\sigma_{rr}^j}{\mu_j}=\left\{\chi^j B_n-\frac{u^j}{r}\left[(n+1)B_{n+1}+(n-1)B_{n-1}\right]-\right.$$
$$\left.-\frac{sw^j}{2}\left[B_{n+2}-B_{n-2}\right]\right\}\begin{Bmatrix}\cos n\theta\\ -\sin n\theta\end{Bmatrix},$$

$$\frac{\sigma_{r\theta}^j}{\mu_j}=\left\{\frac{su^j}{2}\left[B_{n-2}-B_{n+2}\right]-\frac{sw^j}{2}\left[B_{n+2}+B_{n-2}\right]\right\}\begin{Bmatrix}\sin n\theta\\ \cos n\theta\end{Bmatrix},$$

$$\sigma_{\theta\theta}^j=\mu_j\left\{p^j B_n+\frac{su^j}{2}\left[B_{n-2}+B_{n+2}\right]+\right.$$
$$\left.+\frac{sw^j}{2}\left[B_{n+2}-B_{n-2}\right]\right\}\begin{Bmatrix}\cos n\theta\\ -\sin n\theta\end{Bmatrix},$$

$$\sigma_{\theta z}^j=\mu_j\left\{\tau^j\frac{n}{sr}B_n-\frac{dw^j}{dz}B_n'\right\}\begin{Bmatrix}\sin n\theta\\ \cos n\theta\end{Bmatrix},$$

$$\frac{\sigma_{rz}^j}{\mu_j}=\left\{-\tau^j B_n'+\frac{dw^j}{dz}\frac{n}{sr}B_n\right\}\begin{Bmatrix}\cos n\theta\\ -\sin n\theta\end{Bmatrix},$$

$$\sigma_{zz}^j=\mu_j\sigma^j B_n\begin{Bmatrix}\cos n\theta\\ -\sin n\theta\end{Bmatrix},$$

$$\chi^j\equiv\beta_j\frac{dv^j}{dz}+\alpha_j su^j,\ \tau^j\equiv\frac{du^j}{dz}-sv^j,$$

$$p^j\equiv\beta_j\frac{dv^j}{dz}+\gamma_j su^j,\ \sigma^j\equiv\alpha_j\frac{dv^j}{dz}+\beta_j su^j.$$

The equation (6) at $z=z_j$ acquire the form

$$u^{j-1}=u^j\,,\ v^{j-1}=v^j\,,\ w^{j-1}=w^j\,, \qquad (16)$$

$$\frac{\sigma^{j-1}}{\mu_j}=\frac{\sigma^j}{\mu_{j-1}},\ \frac{\tau^{j-1}}{\mu_j}=\frac{\tau^j}{\mu_{j-1}},\ \frac{d}{dz}\left\{\frac{w^{j-1}}{\mu_j}-\frac{w^j}{\mu_{j-1}}\right\}=0. \quad (17)$$

In what follows the homogeneous boundary conditions on the face $z^-=z_1$ (HBCF) mean one of the following forms:

$$\sigma^1\left(z^-\right)=\tau^1\left(z^-\right)=\frac{dw^1\left(z^-\right)}{dz}=0 \qquad \text{(stress free surface)},$$

$$u^1\left(z^-\right)=v^1\left(z^-\right)=w^1\left(z^-\right)=0 \qquad \text{(clamped surface)},$$

$$v^1\left(z^-\right)=0\,,\ \tau^1\left(z^-\right)=\frac{dw^1\left(z^-\right)}{dz}=0 \qquad \text{(mixed conditions 1)},$$

$$\sigma^1\left(z^-\right)=0\,,\ u^1\left(z^-\right)=v^1\left(z^-\right)=0 \qquad \text{(mixed conditions 2)}.$$

The similar formulations are used for $z^+=z_{N+1}$. Any combination of HBCF on $z=z^{\mp}$ and Eqs.(16), (17) give us a system of equations wrt $A_{P,S}^{\pm j},B_S^{\pm j}$ whose $6N\times 6N$ matrix $\mathbf{L}$ yields the frequency equation

$$\det\mathbf{L}=0\,,\qquad \mathbf{L}=\begin{bmatrix}\mathbf{L}_\Delta & 0\\ 0 & \mathbf{L}_\delta\end{bmatrix}. \qquad (18)$$

The important fact is the independency of the frequency equation of the number n and its coincidence with the frequency equation to the respective in-plane or out-of-plane problem with matrix blocks $\mathbf{L}_\Delta$ and $\mathbf{L}_\delta$. Assume that the frequency Eq.(18) has simple roots. Thus these roots can be subdivided into two subsets $s_l\in S_\Delta\cup S_\delta$ due to the polarization of the displacements:

$$s\in S_\Delta:\ w^j=0\,;\ u^j,v^j\neq 0\,, \qquad (19)$$

$$s\in S_\delta:\ u^j=v^j=0\,;\ w^j\neq 0\,. \qquad (20)$$

In addition the frequency equation is symmetrical with respect to $\mp s$, and in case of pure elasticity wrt s and $\bar{s}$. Setting, for the definitiveness, $M_n^l=A_S^{+1}$ or $M_n^l=B_S^{+1}$ the constants $A_{P,S}^{\pm j}$, $B_S^{\pm j}$ are expressed from the equations

$$\mathbf{L}_\Delta\times\left[A_{P,S}^{\pm j}\right]^T=0\,,\ \mathbf{L}_\delta\times\left[B_S^{\pm j}\right]^T=0\,.$$

4 Orthogonality relations

Let us introduce the scalar products across j th layer of any functions f_l^j and g_m^j related to the wave numbers s_l and s_m

$$\left(f_l^j,g_m^j\right)\equiv\int_{z_j}^{z_{j+1}}f_l^j g_m^j dz\,,$$

and compose the following quantities

$$W_{lm}^*\equiv\sum_j\mu_j\left\{\left(\chi_l^j,v_m^j\right)-\left(\tau_m^j,v_l^j\right)\right\}\quad\left(s_l,s_m\in S_\Delta\right),$$

$$G_{lm}^*\equiv\sum_j\mu_j\left\{\left(p_l^j,w_m^j\right)-\left(v_l^j,\frac{d}{dz}w_m^j\right)+\frac{s_l^2-s_m^2}{s_l}\left(u_l^j,w_m^j\right)\right\}$$

$$\left(s_l\in S_\Delta,s_m\in S_\delta\right),$$

$$T_{lm}^{*}=\sum_{j}\mu_j\left(w_l^j,v_m^j\right)\quad\left(s_l,s_m\in S_\delta\right).$$

Consider the cylinders $\Omega_j=\{r\le R,\ z_j\le z\le z_{j+1}\}$ with the lateral and top/bottom surfaces $\Omega_R^j=\{r=R,\ z_j\le z\le z_{j+1}\}$ and $\Omega_{\mp}^j=\{r\le R,\ z=z_{j,j+1}\}$. Let us write the integrals over Ω_R^j

$$\left\langle\mathbf{f}^l,\mathbf{g}^m\right\rangle\equiv\sum_{j}\iint_{\Omega_R^j}f_l^j g_m^j dA=R\sum_j\int_0^{2\pi}\left(f_l^j,g_m^j\right)d\theta .$$

After some simplifications, we obtain for the homogeneous waves the following set of identities ($\xi_n\equiv 1+\delta_n^0$, δ_β^α is a Kronecker delta)

$$\begin{aligned}&\left\langle\sigma_{rr}^l,\mathbf{u}_r^m\right\rangle+\left\langle\sigma_{r\theta}^l,\mathbf{u}_\theta^m\right\rangle+\left\langle\sigma_{rz}^l,\mathbf{u}_z^m\right\rangle-\\&-\left\langle\sigma_{rr}^m,\mathbf{u}_r^l\right\rangle-\left\langle\sigma_{r\theta}^m,\mathbf{u}_\theta^l\right\rangle-\left\langle\sigma_{rz}^m,\mathbf{u}_z^l\right\rangle=\\&=\pi R\xi_n\left\{W_{ml}^{*}B_n(s_mR)B_n'(s_lR)-W_{lm}^{*}B_n'(s_mR)B_n(s_lR)\right\},\end{aligned}\tag{21}$$

$$\begin{aligned}&\left\langle\sigma_{rr}^l,\mathbf{u}_r^m\right\rangle+\left\langle\sigma_{r\theta}^l,\mathbf{u}_\theta^m\right\rangle+\left\langle\sigma_{rz}^l,\mathbf{u}_z^m\right\rangle-\\&-\left\langle\sigma_{rr}^m,\mathbf{u}_r^l\right\rangle-\left\langle\sigma_{r\theta}^m,\mathbf{u}_\theta^l\right\rangle-\left\langle\sigma_{rz}^m,\mathbf{u}_z^l\right\rangle=\\&=\pi\xi_n G_{lm}^{*}\frac{n}{s_m}B_n(s_mR)B_n(s_lR),\end{aligned}\tag{22}$$

$$\begin{aligned}&\left\langle\sigma_{rr}^l,\mathbf{u}_r^m\right\rangle+\left\langle\sigma_{r\theta}^l,\mathbf{u}_\theta^m\right\rangle+\left\langle\sigma_{rz}^l,\mathbf{u}_z^m\right\rangle-\\&-\left\langle\sigma_{rr}^m,\mathbf{u}_r^l\right\rangle-\left\langle\sigma_{r\theta}^m,\mathbf{u}_\theta^l\right\rangle-\left\langle\sigma_{rz}^m,\mathbf{u}_z^l\right\rangle=\\&=\pi R\xi_n T_{lm}^{*}\left(E_{lm}-E_{ml}\right),\end{aligned}\tag{23}$$

$$E_{lm}=\tfrac{1}{2}s_l\left\{B_{n-1}(s_mR)B_{n-2}(s_lR)-B_{n+1}(s_mR)B_{n+2}(s_lR)\right\}.\tag{24}$$

By virtue of the independency of the factors with cylindrical functions B_n we conclude that for $s_l^2\ne s_m^2$

$$W_{lm}^{*}=0,\tag{25}$$

$$G_{lm}^{*}=0,\tag{26}$$

$$T_{lm}^{*}=0.\tag{27}$$

The Eqs.(25)-(27) are the desired OR between modes with the "in-plane" and/or "out-of-plane" polarization. Their physical meaning is the power flow additivity due to the reciprocity. Indeed, substituting $\operatorname{Re}\left\{u_{\alpha l}^j e^{-i\omega t}\right\}$ instead of $u_{\alpha l}^j$ into the averaged power flow

$$P_{rl}^{*}\equiv\frac{\omega}{2\pi}\int_0^{2\pi/\omega}\sum_j\iint_{\Omega_R^j}P_{rl}^j dAdt,$$

$$\left(P_{rl}^j\equiv P_{1l}^j\cos\theta+P_{2l}^j\sin\theta,\ P_{\alpha l}^j=-\sigma_{\alpha\beta l}^j\dot{u}_{\beta l}^j\right)$$

we arrive at the expressions

$$P_r^{*}=\frac{\pi\xi_nR}{2}\operatorname{Re}\left\{\sum_j\mu_j\left[\left(\chi_1^j,\bar{u}_1^j\right)-\left(\bar{\tau}_1^j,v_1^j\right)\right]i\omega\bar{B}_n'(s_lR)B_n(s_lR)\right\}\tag{28}$$

$$\left(s_l\in S_\Delta\right),$$

$$\begin{aligned}P_r^{*}=\frac{\xi_n\pi}{2}\operatorname{Re}\Big\{&\sum_j\mu_j\left(w_1^j,\bar{w}_1^j\right)i\omega\bar{s}_l\left[B_{n-1}(s_lR)\bar{B}_{n-2}(s_lR)-\right.\\&\left.-\bar{B}_{n+2}(s_lR)B_{n+1}(s_lR)\right]\Big\}\qquad\left(s_l\in S_\delta\right),\end{aligned}\tag{29}$$

The choice of $B_n=H_n^{(1,2)}$ in case of pure elasticity for the propagating mode $s_l>0$ yields

$$P_r^{*}=\pm\xi_n c_l W_{ll}^{*},\ c_l=\omega/s_l\quad\left(s_l\in S_\Delta\right),\tag{30}$$

$$P_r^{*}=\pm\xi_n\omega T_{ll}^{*}\quad\left(s_l\in S_\delta\right).\tag{31}$$

$$P_{rl}^{*}\sim\pm c_g^l\left\{K_l+E_l\right\}^{*},\quad R\to+\infty,\quad c_g^l\equiv d\omega/ds\big|_{s=s_l},\tag{32}$$

where K_l, E_l are integrals of the positive kinetic and elastic energy density across the lateral cylindrical surface. The sign $\pm$ is chosen accordingly to the first or second kind of Hankel's function, respectively. Thus, the equations (30)-(32) give us a criterion of the energy propagation to select real s_l, if they exist.

5 Far field of a finite acoustic source

Assume that the laminate motion is caused by an acoustic source in the form of body forces f_α^j, distributed in a finite volume embedded into cylinder $\Omega=\cup\Omega_j$, or in the form of surface load distributed over a finite region on Ω_1^- or Ω_N^+. Another part of the faces satisfies HBCF. Let us represent the laminate response as a function of coordinates r,θ,z and decompose it into Fourier series wrt the angle θ. Upon the general theory of differential equations in partial derivatives the field inside Ω $\left(r<R\right)$ has *two* components: a particular solution due to the acoustic source and a general *homogeneous* solution. At $r>R$ the particular solution vanishes and the field equals the series of modes (7)-(9) for with respective $B_n=H_n^{(1)}$ (or $H_n^{(2)}$) due to the energy radiation principle.

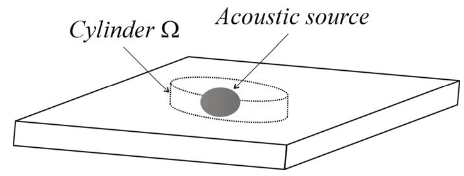

Fig.2 Acoustic source.

On the surfaces Ω_R^j $\left(r=R\right)$ the inner and outer solutions satisfy the continuity of u_α^j and $\sigma_{r\alpha}^j$. For each wave number s_m introduce also a standing wave with $B_n(s_mr)\equiv J_n(s_mr)$ and the same components u_m^j, v_m^j and w_m^j. Then, integrating the total field $\sigma_{\alpha\beta}^j$, u_α^j with this standing wave over cylinder Ω we obtain

$$\sum_j\iint_{\Omega_R^j}\left\{\sigma_{\alpha\beta}^j u_{\alpha m}^j-\sigma_{\alpha\beta m}^j u_\alpha^j\right\}n_\beta dA=\Gamma_{mn},\tag{33}$$

$$\Gamma_{mn} \equiv -\sum_j \iiint_{\Omega^j} \left\{ f_\alpha^j u_{\alpha m}^j \right\} dV + \\ + \left\{ \iint_{\Omega_1^-} + \iint_{\Omega_N^+} \right\} \left\{ \sigma_{\alpha\beta m} u_\alpha - \sigma_{\alpha\beta} u_{\alpha m} \right\} n_\beta dA. \quad (34)$$

Here Γ_{mn} does not contain any unknowns. For example, if the source is given by the stresses $\sigma_{\alpha z}^-$ on Ω_1^- and $\sigma_{\alpha z}^+$ on Ω_N^+ this expression yields

$$\Gamma_{mn} = -\iint_{\Omega_N^+} \left\{ \sigma_{zz}^+ u_{zm}^N + \sigma_{zr}^+ u_{rm}^N + \sigma_{z\theta}^+ u_{\theta m}^N \right\} dA - \\ - \iint_{\Omega_1^-} \left\{ \sigma_{zz}^- u_{zm}^1 + \sigma_{zr}^- u_{rm}^1 + \sigma_{z\theta}^- u_{\theta m}^1 \right\} dA. \quad (35)$$

Then we do the following: replace the field on the lateral surfaces Ω_R^j by the mode series for the outer zones with Hankel's functions $H_n^{(1)} = J_n + iN_n$ (or $H_n^{(2)}$); annihilate in the left hand side of Eq.(33) all waves except $s = s_m$ by taking into accounts OR (25)-(27); simplify the right hand sides in (21)-(23) using the property of cylindrical function

$$J_{n+1}(s_l R) N_n(s_l R) - J_n(s_l R) N_{n+1}(s_l R) = 2/\pi R s_l .$$

Finally it yields the exact mode coefficients

$$\begin{cases} M_n^m = -is_m \Gamma_{mn} / \left\{ 2\xi_n W_{mm}^* \right\}, & s_m \in S_\Delta, \\ M_n^m = -i\Gamma_{mn} / \left\{ 2\xi_n T_{mm}^* \right\}, & s_m \in S_\delta. \end{cases} \quad (36)$$

in the similar form for the combination of trig functions $\cos n\theta$, $\sin n\theta$ or $-\sin n\theta$, $\cos n\theta$. Hence, we suggest a general method to evaluate the "far" field – but in fact the total field at the distance $r > R$, where $2R$ is the longitudinal size of an acoustic source. The method requires the calculation of spectra S_Δ and S_δ, modes (7)-(9) and coefficients (36) in the double series wrt n and s_m. In the case of pure elasticity the classical *far field* as waves propagating to infinity is expressed by ordinary series of n with a finite set of real wave numbers s_m at each frequency.

6 Some exact solutions

Consider a few examples of calculating Γ_{mn}. Assume that the load is distributed over a circular region Ω_N^+ and the surface stresses $\sigma_{zz}^+(r,\theta)$, $\sigma_{rz}^+(r,\theta)$ and $\sigma_{\theta z}^+(r,\theta)$ are expanded into the trigonometrical Fourier series wrt θ. In accordance with the representations (17)-(23) let us for a moment denote coefficients of $\cos n\theta$ (or $-\sin n\theta$) for σ_{zz}^+ and σ_{rz}^+ by $\tau_{zn}^+(r)$ and $\tau_{rn}^+(r)$, respectively. For $\sigma_{\theta z}^+$ the coefficient of $\sin n\theta$ (or $\cos n\theta$) is denoted by $\tau_{\theta n}^+(r)$. The substitution into (123) yields

$$\Gamma_{mn} = -\pi\xi_n \left\{ u_m^N(z^+) T_r^+ + w_m^N(z^+) T_\theta^+ + v_m^N(z^+) T_z^+ \right\},$$

$$T_{r,\theta}^+ = \tfrac{1}{2} \int_0^R \left\{ \left[\tau_{rn}^+(r) + \tau_{\theta n}^+(r) \right] J_{n+1}(s_m r) \pm \left[\tau_{rn}^+(r) - \tau_{\theta n}^+(r) \right] J_{n-1}(s_m r) \right\} r dr ,$$

$$T_z^+ = \int_0^R \tau_{zn}^+(r) J_n(s_m r) r dr .$$

In particular, for a constant normal load $\tau_{z0}^+/2$ we obtain

$$\Gamma_{m0} = -\frac{\pi}{s_m} R J_1(s_m R) \tau_{z0}^+ \times \begin{cases} v_m^N(z^+), & s_m \in S_\Delta \\ 0, & s_m \in S_\delta \end{cases},$$

and for a constant tangent load τ_{10}^+ in the direction x_1 the coefficients are

$$\Gamma_{m1} = \frac{\pi}{s_m} R J_1(s_m R) \tau_{10}^+ \times \begin{cases} u_m^N(z^+), & s_m \in S_\Delta \\ -w_m^N(z^+), & s_m \in S_\delta \end{cases}.$$

Other $\Gamma_{mn} = 0$. It is also easily to obtain the laminate response to a concentrated load. For the concentrated body forces $f_\alpha^j = T_0 \delta_\alpha^\beta \delta(x_1, x_2, x_3 - z_0)$ ($z_j \le z_0 \le z_{j+1}$) at any HBCF we obtain

$$\Gamma_{mn} = -\sum_j \iiint_{\Omega^j} \left\{ f_\alpha^j u_{\alpha m}^j \right\} dV = -T_0 u_{\beta m}^j \Big|_{r=0, z=z_0} ,$$

with a similar result for the concentrated surface load $\sigma_{\alpha z}^+ = \tau_0^+ \delta_\alpha^\beta \delta(x_1, x_2)$:

$$\Gamma_{mn} = -\tau_0^+ u_{\beta m}^N \Big|_{r=0, z=z^+} .$$

Note that these formulae are non singular since the dummy displacements $u_{\beta m}^j(r,\theta,z_0)$ contain Bessel's function $B_n = J_n$ whose value at the origin is regular. However the mode series might have singularity at the origin due to the Hankel functions involved. By the same reason for the transversal load (axisymmetrical problem $\beta = 3$) the terms with $n \ge 1$ vanish and only $\Gamma_{m0} \ne 0$. For the longitudinal load $(\beta = 1,2)$ only $\Gamma_{m1} \ne 0$.

7 Some generalizations

<u>*First*</u> natural generalization is for a fluid loaded laminate. Assume that some layers are not solids but ideal compressible (or incompressible) fluids. Thus, in each fluid marked by zero we must satisfy the equation ($P^0 = -\lambda_0 \nabla \mathbf{U}^0$ is a pressure)

$$-\nabla P^0 + \rho_0 \omega^2 \mathbf{U}^0 + \mathbf{f}^0 = 0 ,$$

the continuity condition for normal displacements between solid and fluid, and the condition for normal stress equals opposite pressure on the interface. The analogues of HBCF in case of the facial fluid surface are the absence of pressure or of the normal displacements. The displacement vector in fluid is determined similarly to (7)-(9) with $w^j = 0$ and with pressure

$$P^0 = \lambda_0 p^0(z) B_n(sr) \begin{Bmatrix} \cos n\theta \\ -\sin n\theta \end{Bmatrix}$$

$$\left(p^0 \equiv -k_0^2 s^{-1} u^0, k_0 \equiv \omega / c_0, c_0 \equiv \sqrt{\lambda_0/\rho_0} \right).$$

The waves with the "out-of-plane" polarization in the laminate remain unperturbed, but the "in-plane" waves have some corrections. The identities (22)-(23) and orthogonality relations (26)-(27) remain in force. The formulae (21) and (25) must use W_{lm}^* corrected as follows

$$W^*_{lm} \equiv \sum_j \mu_j \left\{ \left(\chi^j_l , v^j_m \right) - \left(\tau^j_m , v^j_l \right) \right\} - \sum_k \lambda_{0k} \left(p^{0k}_l , u^{0k}_m \right) \quad (37)$$

$$\left(s_l , s_m \in S_\Delta \right),$$

where number k corresponds to fluid layers. The formulae (36) also remain in force but Γ_{mn} should have the additional volume integrals and replaced facial integrals if these faces are of fluid ply given by the following terms

$$-\sum_k \iiint_{\Omega^0_k} \left\{ f^{0k}_\alpha u^{0k}_{\alpha m} \right\} dV , \quad \iint_{\Omega^{N_0}_+} P^+ u^{N_0}_{zm} dA , \quad \iint_{\Omega^{1_0}_-} P^- u^{1_0}_{zm} dA .$$

The <u>*second*</u> generalization concerns the layers of possibly infinite thicknesses. Now the spectrum of the respective boundary value problem is subdivided into discrete part $S_\Delta \cup S_\delta$, whose homogeneous waves are described similarly to Section 3 (trig functions are replaced by exponents for infinite thickness), and by continuous part $\eta = \eta_\Delta \cup \eta_\delta$ for which we obtain

$$\begin{bmatrix} u^j_r \\ u^j_\theta \\ u^j_z \end{bmatrix} = \sum_{n=0}^{+\infty} \int_\eta \begin{bmatrix} \left(-u^j B'_n + w^j nB_n / sr \right) \left(M^c_n \cos n\theta - M^s_n \sin n\theta \right) \\ \left(u^j nB_n / sr - w^j B'_n \right) \left(M^c_n \sin n\theta + M^s_n \cos n\theta \right) \\ v^j B_n \left(M^c_n \cos n\theta - M^s_n \sin n\theta \right) \end{bmatrix} ds,$$

$$M^{c,s}_n = M^{c,s}_n (s) .$$

The continuous part consists of the cutoffs for radicals $q_{P,S}$ in each half space. It is important that for the case of a finite source the field of continuous part satisfies the homogenous equations at $r > R$. By this reason the identities (21)-(23) hold not only for a discrete part of spectrum but also when $s_l \in S_{\Delta,\delta}$ and $s_m \in \eta_{\Delta,\delta}$ $\left(s^2_l \neq s^2_m \right)$ or vice versa. The right hand side in (21)-(23) must be integrated over $\eta_{\Delta,\delta}$. Thus, the relations (25)-(27) are valid and

- Different homogeneous waves of discrete spectrum are orthogonal to each other;
- Homogenous waves of discrete spectrum are orthogonal to waves of continuous spectrum.

This immediately results in the mode coefficients (36) for $s_m \in S_\Delta \cup S_\delta$ omitting the consideration of the direct and inverse Fourier transform. The necessary wave numbers is easily obtained from the frequency equation for the plane problem in laminate. However, for the continuous part of spectrum there is no simplification in the consideration of the cutoffs in Fourier integrals.

8 Conclusion

The obtained results can be clearly subdivided into *three* groups. *First* group includes orthogonality relations for the cylindrical guided waves satisfying homogeneous boundary conditions on the laminate faces. They correlate with the results of previous authors for an elastic layer and plane waves, which can be obtained as a limit case for large radius. The explicit expressions for reciprocity relations are obtained as well for both elastic and linearly viscoelastic media due to the symmetry of their energy functional. The *second* group describes solving methods for the important problem to evaluate the far field of an acoustic source - surface loads or body forces localised in a finite region – which can be solved in a closed form. The obtained Green's functions can be used to represent a field of arbitrary aperture by convolution integrals. The solution for a circular region is of interest for modelling circular transducers. In particular, having the time harmonic field radiated into laminate we may also evaluate a pulse train using harmonic synthesis. The *third* group generalizes our results for the case of fluid loaded laminate and/or layers with infinite thicknesses, for which we obtain the closed form of 3D Rayleigh, Love, Stonely or Scholte waves. As far as the question of the guided wave completeness is concerned, we may refer to the more general result. Normally, the total set of eigenfunctions of the polynomial operator pencil has multiple completeness (accordingly to its degree) in the functional Sobolev's space on a cross-section of the geometrical region considered (see [11]). Reducing this set we arrive at the ordinary completeness, e.g., for basic functions $B_n = H^{(1)}_n$ the subset $\operatorname{Im} s_l < 0$ is excluded. The same property is expected for 2D and 3D guided waves in laminates.

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