An overview of models for viscothermal wave propagation, including fluid structure interaction

Willem Beltman[a], Ysbrand Wijnant[b] and Marten Nijhof[a]

[a]Intel Corporation, 2111 NE 25th Avenue, M/S JF2-86, Hillsboro, OR 97124, USA
[b]University of Twente, Dept. Mechanical Engineering, P.O. Box 217, 7500 AE Enschede, Netherlands
willem.m.beltman@intel.com

In acoustics, the standard wave propagation models neglect the effects of viscosity and thermal conductivity. When waves propagate in narrow tubes or thin layers these simplifications might not be accurate. This paper presents an overview of models that take into account the effects of inertia, viscosity, thermal conductivity and compressibility. Based on the use of dimensionless parameters, three classes of models are outlined. The most important dimensionless parameter is the shear wave number, an unsteady Reynolds number that indicates the ratio between inertial and viscous effects. These viscothermal wave propagation models can be coupled to structural models to capture the fluid structure interaction. Analytical solutions can be found for these coupled scenarios for simple geometries and boundary conditions. For more complex geometries, numerical models were developed. Examples of applications of these models are also presented.

1   Introduction

When acoustic waves propagate in thin layers or narrow tubes, viscous and thermal effects may become important. These effects are typically neglected in the standard wave equation, which assumes adiabatic and inviscid behavior. The propagation of sound waves has been the topic of extensive studies [1-5]. Models were developed that include viscosity, inertia and thermal conductivity. In addition, models were developed that describe the propagation of sound waves in thin layers between for example parallel surfaces [6-13]. Analytical solutions for the resulting equations can only be found for simple boundary conditions and geometries. For more complex geometries, numerical solution techniques were developed, for example finite element models. The motion of the gas or fluid on the interface can be coupled to the structural motion of vibrating structures, thereby establishing fully coupled fluid structure interaction conditions. Applications of these types of models have included for example the wave propagation in thin tubes [1-5] or spherical resonators [14-16], the wave propagation in narrow channels in inkjet print heads including the elastic effect of the print head itself [35], the mechanical behavior of microphones backed by a thin air layer [5,7,8,22], the behavior of MEMS devices [19,33,34] and models of the wave propagation in the human cochlea [28]. In the paper, first the theory is discussed, followed the analytical and numerical solution methods, including a new 3D viscothermal model, and an overview of applications.

2   Viscothermal wave propagation

This paper deals with linear, small perturbations of a fluid at rest. The expressions for the velocity vector v, the temperature T, the pressure p and the density ρ are:

$$\bar{v} = c_0 v \cdot e^{i\omega t} \quad ; \quad \bar{T} = T_0 \left[1 + T \cdot e^{i\omega t} \right]$$

$$\bar{p} = p_0 \left[1 + p \cdot e^{i\omega t} \right] \quad ; \quad \bar{\rho} = \rho_0 \left[1 + \rho \cdot e^{i\omega t} \right] \tag{1}$$

For non linear effects in viscothermal fluids, see [17]. Temperature gradients can also induce acoustic oscillations [18], but that is beyond the scope of the present paper.

2.1   Basic equations

The basic equations for viscothermal wave propagation are the Navier Stokes equations, the equation of continuity, the equation of state and the energy equation. When the expressions for the basic variables are inserted into these equations, the following expressions are obtained:

$$iv = -\frac{1}{k\gamma} \nabla p + \frac{1}{s^2} \left(\frac{4}{3} + \xi \right) \nabla (\nabla \cdot v) - \frac{1}{s^2} \nabla \times (\nabla \times v)$$

$$\nabla \cdot v + ik\rho = 0$$

$$p = \rho + T \tag{2}$$

$$iT = \frac{1}{s^2 \sigma^2} \Delta T + i \left[\frac{\gamma - 1}{\gamma} \right] p$$

In these expressions, dimensionless parameters were introduced, see section 2.2 below. A characteristic length scale, l, was introduced in this process, for example representing the tube radius or layer thickness [4]

2.2   Dimensionless parameters

The following dimensionless parameters are relevant for viscothermal wave propagation:

$$s = l \sqrt{\frac{\rho_0 \omega}{\mu}} \quad ; \quad k = \frac{\omega l}{c_0} \quad ; \quad \gamma = \frac{C_p}{C_v}$$

$$\sigma = \sqrt{\frac{\mu C_p}{\lambda}} \quad ; \quad \xi = \frac{\eta}{\mu} \tag{3}$$

The most important parameters are the shear wave number, s, and the reduced frequency, k. The shear wave number is an unsteady Reynolds number that represents the ratio between inertia and viscosity. For high values of s the viscous effects can be neglected, whereas for low values of s the viscous effects dominate. The reduced frequency, k, represents the ratio between the length scale l and the acoustic wavelength. For small values of k the length scale is small compared to the characteristic length scale. The parameters ξ, σ and γ are material dependent.

2.3   Full linearized Navier Stokes model

The first model is a solution to the full set of equations, as presented in (3). After some algebra, the solution can be written as:

$$\left[\Delta + k_a^2 \right] \left[\Delta + k_h^2 \right] = 0 \tag{4}$$

Where k_a and k_h are the acoustic and entropic wave numbers according to:

$$k_a = \frac{2k^2}{C_1 + \sqrt{C_1^2 - 4C_2}} \; ; k_h = \frac{2k^2}{C_1 - \sqrt{C_1^2 - 4C_2}}$$

$$C_1 = \left[1 + \frac{ik^2}{s^2} \left[\left(\frac{4}{3} + \xi \right) + \frac{\gamma}{\sigma^2} \right] \right] \qquad (5)$$

$$C_2 = \frac{ik^2}{s^2 \sigma^2} \left[1 + \frac{i\gamma k^2}{s^2} \left(\frac{4}{3} + \xi \right) \right]$$

The other quantities can be calculated from T_a and T_h [4].

2.4 Simplified Navier Stokes models

In this class of models the effects of compressibility or thermal conductivity are neglected [9]. It can be shown that the full dimensionless parameter range can be covered by the Full Linearized Navier Stokes model, the low reduced frequency model and the standard wave equation. This implies that the simplified models contain inconsistent assumptions.

2.5 Low reduced frequency model

The low reduced frequency model, first published in [1], introduced some assumptions that lead to a simple yet very powerful model. The following simplifications are introduced:

- the acoustic wave length is large compared to the length scale l: $k \ll 1$
- the acoustic wave length is large compared to the boundary layer thickness: $k/s \ll 1$

When these assumptions are incorporated into the basic equations (3), they can be significantly simplified. In this model, the assumptions lead to a constant pressure across the layer thickness or tube cross section. The resulting equation can be written as:

$$\Delta^{pd} p - k^2 \Gamma^2 p = -ikn\Gamma^2 \Re \qquad (6)$$

The Laplace operator Δ^{pd} now only contains partial derivatives with respect to the propagation directions, i.e. the in plane coordinates for layers, and the axial direction for tubes. The propagation constant Γ is given by:

$$\Gamma = \sqrt{\frac{\gamma}{nB}} \qquad (7)$$

The quantity n is the polytropic constant, and depends on the product $s\sigma$. The function B depends on the shear wave number s and describes the velocity profile across the layer thickness or tube cross section. For low shear wave numbers the velocity profile, determined by the function B, approached the parabolic viscous profile, whereas for high shear wave numbers it approaches the constant velocity from the wave equation.

3 Fluid structure interaction

The expression $\Re$ on the right hand side in (6) is a source term that is a function of the normal velocity at the fluid structure interface. It describes the pressure disturbances that are introduced by squeeze effect of the boundaries. This term therefore establishes the acousto-elastic coupling for the fluid structure interaction. The viscothermal model can be easily coupled to the vibrations of structural elements such as membranes, plates and elastic solids. It should be noted that typically a no slip condition is used at the fluid structure interface. There are also situations, for example rarefied gases, where jump conditions are more appropriate, see [19].

4 Analytical solutions

Analytical solutions for the models outlined in section 2 can be found for simple geometries and boundary conditions. For example, the full Navier Stokes model has been solved analytically for circular tubes, the spherical resonator and a circular membrane backed by a thin gas layer [5]. The low reduced frequency model can be solved analytically for a number of cases because of its relative simplicity. However, for more complex geometries, boundary conditions and in order to capture the fluid structure interaction for the more general case, numerical techniques were developed. An overview of these techniques will be presented in the next section.

5 Numerical techniques

5.1 Literature

Several numerical solution techniques for viscothermal wave propagation, including fluid structure interaction, are described in the literature. In [20] an integral formulation is presented for viscothermal wave propagation, based on a Full Linearized Navier Stokes model. The numerical solution for a thin cylindrical layer was compared to the analytical solution to validate the model. The model was then extended in to include the fluid structure interaction with vibrating membranes [21-23] and plates [24]. The formulation leads to:

$$\begin{bmatrix} \mathbf{Z}_s(\omega) & \mathbf{C}(\omega) \\ \mathbf{C}^T(\omega) & \mathbf{D}(\omega) \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{U} \\ \mathbf{P} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ 0 \end{Bmatrix} \quad (8)$$

Where $\mathbf{Z}_s(\omega)$ is the matrix of the membrane impedance that is influence by the viscothermal fluid, $\mathbf{D}(\omega)$ is the matrix of admittance of the fluid, $\mathbf{C}(\omega)$ is the coupling matrix between the nodal displacement vector $\mathbf{U}$ of the membrane and the nodal pressure degrees of freedom $\mathbf{P}$. The external excitation on the membrane is given by $\mathbf{F}$. Note the frequency dependent nature of the matrices involved. A

modal superposition was then used to describe the structural displacements. Similar models were used in [26], and a hybrid analytical/numerical approach was used in [25].

5.2 Low reduced frequency model

The low reduced frequency model resembles the standard wave equation, with a complex and frequency dependent speed of sound, and a forcing term representing the coupling at the fluid structure interface. A finite element approach for this problem is relatively straightforward and leads to [27]:

$$\begin{bmatrix} \mathbf{K}^s - \omega^2 \mathbf{M}^s & -\mathbf{K}^c \\ -\omega^2 \mathbf{M}^c(s) & \mathbf{K}^a - \omega^2 \mathbf{M}^a(s) \end{bmatrix} \cdot \begin{Bmatrix} \mathbf{U} \\ \mathbf{P} \end{Bmatrix} = \begin{Bmatrix} \mathbf{F} \\ 0 \end{Bmatrix} \quad (9)$$

This system contains the mass matrices and the stiffness matrices of the structural part and the acoustic part, $\mathbf{M}^s$, $\mathbf{K}^s$, $\mathbf{M}^a$ and $\mathbf{K}^a$ respectively. Note the frequency dependent nature of $\mathbf{M}^a$ due to the frequency dependent speed of sound. The coupling is established by the two coupling matrices, $\mathbf{M}^c(s)$ and $\mathbf{K}^c$, which are related as:

$$\mathbf{M}^c(s) = C(s) \cdot \mathbf{K}^{cT} \quad (10)$$

Where *C(s)* is a shear wave dependent multiplication term [27]. The matrix $\mathbf{M}^c(s)$ only differs from the standard coupling matrix due to this pre multiplication factor. Note that the system of equations given in (9) is asymmetric, but it can be made symmetric by dividing the second set of equations by the pre multiplication factor *C(s)*.

5.3 New 3D viscothermal model

For complex geometries, a 'small' length scale is difficult to identify. Also, transitions between areas with different cross sections results in 3D effects. Therefore, the low reduced frequency model can not describe the wave behavior correctly. Therefore, a more general approach was developed to solve the full linearized system of equations. The resulting viscothermal finite element was implemented in the finite element package Comsol[TM] [38]. The starting equations are now written as, see (2):

$$-\nabla \cdot \boldsymbol{\tau} + is^2 \mathbf{v} + \frac{s^2}{k\gamma} \nabla p = 0$$

$$\Delta T - is^2\sigma^2 T - is^2\sigma^2 \left[\frac{\gamma-1}{\gamma}\right] p = 0 \quad (11)$$

$$\nabla \cdot \mathbf{v} - ikT + ikp = 0$$

where the viscous stress tensor $\boldsymbol{\tau}$ is given by:

$$\boldsymbol{\tau} = \xi(\nabla \cdot \mathbf{v})\mathbf{I} + \left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T\right) \quad (12)$$

The equation of state was used to remove the density from the equations. Hence, once the solution is known, the acoustic density follows from:

$$\rho = p - T \quad (13)$$

Applying Green's theorem, the weak formulation of the equations yields:

$$\int_\Omega \boldsymbol{\tau} \cdot \left(\nabla \mathbf{v}^{tf}\right) + is^2 \mathbf{v} \cdot \mathbf{v}^{tf} - \frac{s^2}{k\gamma} p\left(\nabla \cdot \mathbf{v}^{tf}\right) d\Omega =$$

$$\int_\Gamma (\boldsymbol{\sigma} \cdot \mathbf{n}) \cdot \mathbf{v}^{tf} d\Gamma$$

$$\int_\Omega \nabla T \cdot \nabla T^{tf} + is^2\sigma^2 T \cdot T^{tf} - is^2\sigma^2 \left[\frac{\gamma-1}{\gamma}\right] pT^{tf} d\Omega$$

$$= \int_\Gamma \left((\nabla T) \cdot \mathbf{n}\right) \cdot T^{tf} d\Gamma$$

$$\int_\Omega (\nabla \cdot \mathbf{v} - ikT + ikp) p^{tf} d\Omega = 0$$

(14)

where the superscript *tf* denotes a test function and $\boldsymbol{\sigma} = \boldsymbol{\tau} - p\mathbf{I}$ denotes the total stress tensor. Consequently, the natural boundary conditions are the total stress tensor $\boldsymbol{\sigma} \cdot \mathbf{n}$ and the heat flux $\nabla T \cdot \mathbf{n}$ which need to be specified on the enclosing boundary Γ. It is noted that this formulation can lead to instable behavior if the order of the approximate fields are equal for all values. One can use so-called 'bubble functions' to obtain a stable solution [36,37]. In the current implementation, quadratic shape functions for the velocity and a linear shape function for the temperature and pressure were used to obtain a stable solution. Although not reported here, the implementation was validated using the low reduced frequency model.

6 Applications

As an illustration of the new element, results are presented for a wave traveling through a complex geometry with strong 3D characteristics, as shown in Figure 1. At the left face, a unit pressure is assumed and an impedance boundary condition $\rho_0 c_0$ is assumed at the right face. Temperature conditions are adiabatic on this face. The remaining faces have been assumed to be rigid (isothermal) walls. The outer dimension of the geometry is 0.5x0.5x0.5 cm. The frequency is 300 Hz. The results are given in Figures 2 and 3, in which iso-surfaces of the pressure and the acoustic velocity are shown.
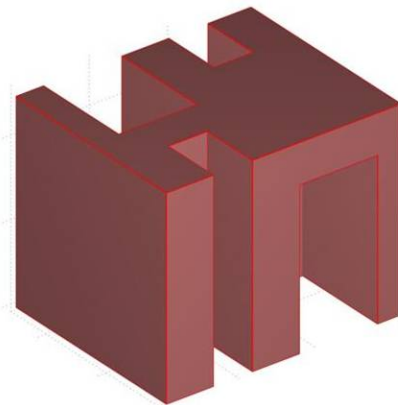

Figure 1. The analyzed geometry.

Compared to the solution of the standard wave equation, the acoustic pressure reduces by more than a factor of 2.

This demonstrates that for devices of this dimension, viscothermal effects are important. The implemented viscothermal finite element can also be used in coupled fluid structure interaction analysis.
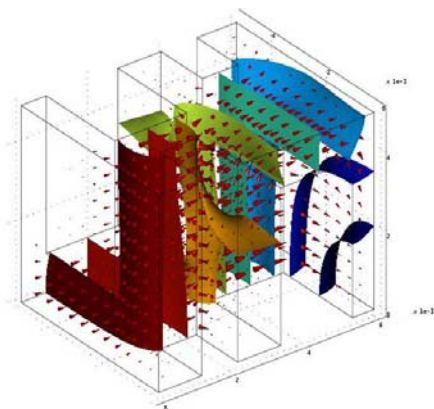

Figure 2. Iso-pressure surface and velocity for the non-viscothermal (standard) case.
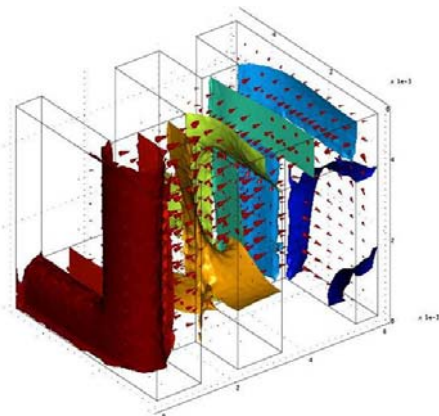

Figure 3. Iso-pressure surface and velocity for the viscothermal case.

These viscothermal models are important in many applications, such as:

- Wave propagation in thin tubes, including the effects of flexible walls [1-5]

- The mechanical behavior of MEMS devices with thin gas or fluid layers [19,33,34]

- The dynamic behavior of inkjet print heads, including the wave propagation in narrow ink channels coupled with the structural motion of the surrounding print head [35]

- The behavior of plates and double wall panels with a thin gas or fluid layer [9-13,24,29,30]

- The behavior of spherical resonators [5,14-16]

- The dynamic behavior of for example microphones, consisting of a membrane backed by a thin air layer [5,7,8,22]

- A model of wave propagation in the cochlea [28]

- The absorption of resonators and tubes [31]

5   Conclusion

An overview was presented of models for viscothermal wave propagation, including fluid structure interaction. In addition, a new finite element model was developed that describes the full 3D viscothermal wave propagation, including fluid structure interaction. An application of the new model was presented. This model is critical in describing viscothermal wave propagation for complex 3D geometries with for example transitions in cross sectional area.

Acknowledgments

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