Acoustics'08 Paris
June 29-July 4, 2008
www.acoustics08-paris.org

Transparent boundary condition for acoustic propagation in lined guide with mean flow

Emmanuel Redon[a], Boureima Ouedraogo[a], Anne-Sophie Bonnet-Ben Dhia[b], Jean-Francois Mercier[b] and Colin Chambeyron[b]

[a]Université de Bourgogne, LRMA, 9, avenue Alain Savary, 21000 Dijon, France
[b]CNRS, ENSTA - 32 Boulevard Victor, 75 015 Paris, France
emmanuel.redon@u-bourgogne.fr

A finite element analysis of acoustic radiation in an infinite lined guide with mean flow is studied. In order to bound the domain, transparent boundary conditions are introduced by means of a Dirichlet to Neumann (DtN) operator based on a modal decomposition. This decomposition is easy to carry out in a hard-walled guide. With absorbing lining, many difficulties occur even without mean flow. Because the eigenvalue problem is no longer selfadjoint, acoustic modes are not orthogonal with respect to the L2-dot product. However, an orthogonality relation exists which permit to write the modal decomposition. For a lined guide with uniform mean flow, orthogonality relation doesn't exist but a new dot product allows us to define the DtN operator. We consider first the case of an infinite rectangular two-dimensional lined guide with uniform mean flow to present the methodology. Then, the extension to the axisymmetric cylindrical problem is presented.

1 Introduction

The study of the propagation of acoustic waves with mean flows remains an open and difficult problem whose applications involve the sectors of aerospace (turbine aircraft) and automobiles (mufflers). For these problems, there are often areas where the geometry is complex, requiring the use of finite element method and areas equivalent to waveguides allowing the use of modal methods. Our work concerns the study of a lined acoustic guide with uniform mean flow. The system is modelled here by the Helmholtz convected equation. The use of the finite element method requires truncating the calculation field. It uses a transparent boundary condition described by a DtN (Dirichlet to Neumann) operator based on a modal decomposition. The case of a two-dimensional guide with absorbent material reveals difficulties even in the absence of flow: the operator is no longer self-adjoint and modes are no longer orthogonal under usual dot product. A new dot product is then introduced to allow the writing of the transparent boundary condition. We attempt here to generalize the results for axisymmetric cylindrical geometries (0rz plane) in order to treat more realistic cases. The research of the modes is made more difficult since they are now expressed by Bessel functions.

2 Two-dimensional guide

2.1 Theoretical models

We consider an infinite two-dimensional duct of height h containing an assumed fixed acoustic source and a fluid in subsonic uniform flow of speed $\boldsymbol{v_0}$ according to $\boldsymbol{e_x}$. The upper wall $\boldsymbol{\Gamma_Z}$ $\boldsymbol{(y = h)}$ is covered with an absorbing material characterized by an impedance $\boldsymbol{Z}$ ($\boldsymbol{Z} \in \mathbb{C}$). The lower horizontal boundary $\boldsymbol{\Gamma}$ of the duct is supposed here perfectly rigid (see figure below). The problem is posed in the $\boldsymbol{Oxy}$ plane where the $\boldsymbol{x}$-axis is parallel to the walls of the guide.

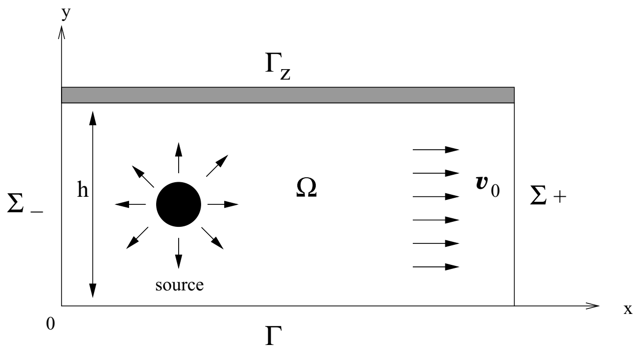


Figure 1: Problem Geometry

2.2 Hard-walled guide

In the absence of flow, the acoustic wave propagation is described by the Helmholtz equation:

$$\boldsymbol{\Delta p + k^2 p = f \quad in \quad \Omega} \tag{1}$$

Where $\boldsymbol{f}$ is a source with compact support. For perfectly rigid walls, the boundary conditions on the border $\boldsymbol{\partial\Omega = \Gamma \cup \Gamma_Z \cup \Sigma_+ \cup \Sigma_-}$ are written as:

$$\frac{\partial p}{\partial n} = 0 \qquad on \quad \Gamma \quad and \quad \Gamma_Z \tag{2}$$

$$\frac{\partial p}{\partial n} = -T^{\pm}(p) \quad on \quad \Sigma_{\pm} \tag{3}$$

where $\boldsymbol{n}$ means the external normal, $\boldsymbol{k = \omega/c}$ the wavenumber with $\boldsymbol{c}$ the speed of sound, and $\boldsymbol{T^-(p)}$ and $\boldsymbol{T^+(p)}$ the "Dirichlet to Neumann" (DtN) operators for the transparent boundary condition at the extremities $\boldsymbol{\Sigma_-}$ and $\boldsymbol{\Sigma_+}$ of the guide. To clarify the DtN operators (3) on the boundaries $\boldsymbol{\Sigma_-}$ and $\boldsymbol{\Sigma_+}$ it is necessary to determine the modes of the guide which are solutions of (1). They are the classical solutions obtained by the variables separation method:

$$p\ (x, y) = \varphi(y) e^{i\beta x} \tag{4}$$

In the particular case of a duct with perfectly rigid walls without flow, we can see(see [2]) that there are two kinds of modes:

$$p_n^{\pm}\ (x, y) = \varphi_n^{\pm}(y) e^{i\beta_n^{\pm} x} \tag{5}$$

where indices $\pm$ correspond to the direction of mode propagation. One easily deduces the distinction between the propagative modes ($\boldsymbol{\beta_n^{\pm}} \in \mathbb{R}$) and the evanescent modes ($\boldsymbol{\beta_n^{\pm}} \notin \mathbb{R}$). It is also noted that: $\boldsymbol{\varphi_n^+(y) = \varphi_n^-(y) = \varphi_n(y)}$ and $\boldsymbol{\beta_n^+ = -\beta_n^-}$. Finally, the modes of the duct can form an orthonormal basis of $\boldsymbol{L^2(\Sigma_S)}$ verifying the boundary conditions: $\boldsymbol{\partial\varphi_n/\partial y = 0}$ for $\boldsymbol{y = 0}$ and $\boldsymbol{y = h}$:

$$\varphi_0 = 1/\sqrt{h}\,;\ \ \varphi_n = \sqrt{2/h}\, cos(\frac{n\pi}{h} y)\ ,\ \ n \geq 1 \tag{6}$$

The constant of propagation $\boldsymbol{\beta_n}$ is given by the equation of dispersion:

$$\beta_n^2 = k^2 - \alpha_n^2 \tag{7}$$

With $\alpha_n = n\pi/h$. The DtN operators on Σ_- and Σ_+ can be written as:

$$T^{\pm}(p) = \mp \sum_{n \geq 0} i\beta_n^{\pm}(p, \varphi_n)_{\Sigma_\pm} \varphi_n(y) \qquad (8)$$

$$(p, \varphi)_{\Sigma_\pm} = \int_{\Sigma_\pm} p\, \overline{\varphi}\, d\Sigma \qquad (9)$$

where $(.,.)$ represents the dot product on $L^2(\Sigma_\pm)$ and $\overline{\varphi}$ the complex conjugate of φ. The associated variational formulation consist to find $p \in H^1(\Omega)$ such as $\forall \psi \in H^1(\Omega)$:

$$\int_\Omega \nabla p . \nabla \bar{\psi}\, d\Omega - k^2 \int_\Omega p\, \bar{\psi}\, d\Omega + \int_{\Sigma_\pm} T^{\pm}(p)\, \bar{\psi}\, d\Sigma = \int_\Omega f\, \bar{\psi}\, d\Omega \qquad (10)$$

2.3 Lined guide without flow

In the presence of absorbent material on a wall (generalization to two treated walls does not raise any difficulties) propagation of acoustic waves is still governed by the Helmholtz equation (1), only the boundary conditions change on Γ_Z:

$$\frac{\partial p}{\partial n} = \frac{ikp}{Z} \quad on \quad \Gamma_Z \qquad (11)$$

The research of the solutions with separate variables leads to the following transcendent equation:

$$-\alpha_n tan(\alpha_n h) = \frac{ik}{Z}$$

For a purely real impedance (ie. pure resistance), as in the case of a perfectly rigid guide, the modes form an orthonormal basis of $L^2(\Sigma_\pm)$ and determining the DtN operator does not pose additional difficulties. On the other hand, for a complex impedance wall the problem is no longer self-adjoint (because the adjoint depends on $\bar{Z}$), modes are no longer orthogonal under usual dot product on (9). However, if we define a new dot product $(p, \varphi_n)^*_\Sigma$, it is possible to establish an orthogonality relationship. The DtN operator is expressed then by:

$$T_Z^{\pm}(p) = \pm \sum_{n \geq 0} i\beta_n^{\pm}(p, \varphi_n)^*_{\Sigma_\pm} \;\; \varphi_n(y) \qquad (12)$$

Where $(.,.)^*$ represents the new dot product difined as:

$$(p, \varphi)^*_{\Sigma_\pm} = \int_{\Sigma_\pm} p\, \varphi\, d\Sigma \qquad (13)$$

There are exceptional values of the impedance Z for which the normalization of the modes (6) is no longer possible, and the operator DtN can no longer be explained. These exceptional values Z_c were deduced from the equation (12) for the values of α_n complex roots of the equation :

$$sin(2\alpha_n h) + 2\alpha_n h = 0 \qquad (14)$$

Apart from the exceptional values Z_c of the impedance, the transverse numbers of waves α_n, solutions to (12),

n	α_n^+	β_n^+
1	0.7897-1.1705i	7.0543+0.1310i
2	2.8012-0.3759i	6.4282+0.1638i
3	6.1213-0.1609i	3.4142+0.2958i
4	9.3179-0.1077i	0.1631+6.1513i
5	12.4865-0.0803i	0.0970+10.340i

Table 1: Roots of $-\alpha_n tan(\alpha_n) = ik/Z$ for the corresponding $Z = 3.5(1+i)$ and β_n, k=7.

are sought by the Newton-Raphson method (see Table 1). The complex impedance value leads to a constant of propagation $\beta_n^{\pm}$ always complex (7) responsible for the attenuation in the guide, due to $\pm Im(\beta_n^{\pm}) > 0$. Nevertheless, as in the case without absorbent, it was still $\varphi_n^+(y) = \varphi_n^-(y)$ and $\beta_n^+ = -\beta_n^-$.

2.4 Lined guide with uniform mean flow

In the presence of a uniform mean flow ($M = v_0/c$) and a treated boundary of impedance Z, the problem in the two-dimensional guide is described by the convected Helmholtz equation:

$$(1 - M^2)\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + 2ikM\frac{\partial p}{\partial x} + k^2 p = 0 \;\; in\ \Omega \qquad (15)$$

$$\frac{\partial p}{\partial n} = 0 \qquad (\Gamma) \qquad (16)$$

$$\frac{\partial p}{\partial n} = -\frac{i}{kZ}(M\frac{\partial}{\partial x} - ik)^2 p \qquad (\Gamma_Z) \qquad (17)$$

$$\frac{\partial p}{\partial n} = -T_{ZM}^{\pm}(p) \qquad (\Sigma_\pm) \qquad (18)$$

Note, on the boundary condition on Γ_Z (17) a tangential derivative of the pressure appears. These terms are at the origin of additional difficulties clarified later on. We are looking for mode solutions of (15) in the form:

$$p_n(x, y) = A_n\, cos(\alpha_n\, y)e^{i\beta_n x} \qquad (19)$$

The dispersion relation is now expressed by :

$$\alpha_n^2 = k^2 - (1 - M^2)\beta_n^2 - 2kM\beta_n \qquad (20)$$

Where α_n is now the solution of an equation which depends on β_n:

$$-\alpha_n tan(\alpha_n h) = \frac{i\,(\beta_n^2 + \alpha_n^2)}{kZ} \qquad (21)$$

One can divide the solutions into two families according to the sign of the imaginary part of β_n. In the situations for which there are not unstable modes (ie. small M and large $|Z|$), the couples α_n, β_n solutions of (20) and (21) are modes which are propagated towards the downstream (respectively upstream) when $Im(\beta_n) > 0$ (resp. $Im(\beta_n) < 0$), and they are then noted : α_n^+, β_n^+ (resp. α_n^-, β_n^-).
Writing the DtN operator requires the definition of a new dot product noted $((.,.))$ which appears naturally in the variational formulation. It is expressed by :

$$((p, \varphi))_{\Sigma_\pm} = \int_{\Sigma_\pm} p\, \varphi\, d\Sigma - \frac{i\, M^2}{kZ(1 - M^2)} p(h)\varphi(h) \qquad (22)$$

The modes are no longer orthogonal (ie. $((\varphi_n, \varphi_m))_{\Sigma_\pm} \neq \delta_{nm}$), and it raises:

$$((\varphi_n, \varphi_m))_{\Sigma_\pm} = O_{nm} \tag{23}$$

where O is a spectral matrix which allows defining the new DtN operator such as:

$$T^{\pm}_{ZM}(p) = \pm \sum_{n,m \geq 0} i\beta^{\pm}_n (O^{-1})_{nm} \, ((p, \varphi_m))_{\Sigma_\pm} \; \varphi_n(y) \tag{24}$$

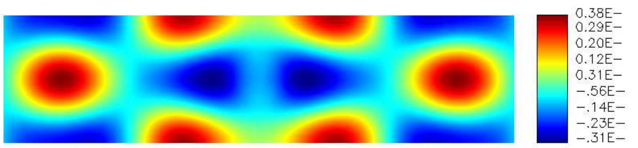
The transparent conditions described by the DtN operator (24) thus require the calculation of the modes, still performed by the Raphson Newton method in the complex plane starting from Equations (20) and (21) (see Table 2). The tangential derivative of the pressure (17) leads to the emergence of an additional term on the border ($y = h$) in (22). The spectral matrix O is not diagonal, its evaluation is thus more expensive than in the last cases where the matrix O is the identity thanks to an orthogonality relation. However, it was verified that the spectral matrix O tends to becoming diagonal when n and m are large.

n	α^+_n	β^+_n
1	0.4820-0.8857i	5.4242+0.0607i
2	2.9328-0.2236i	4.7482+0.1021i
3	6.1445-0.1357i	1.9054+0.2175i
4	9.2440-0.0621i	-2.2007+5.8940i
5	12.4071-0.0014i	-2.3059+10.4875i
n	α^-_n	β^-_n
1	2.1452-1.8204i	-9.9283-0.5631i
2	2.0977-0.6508i	-9.7133-0.2026i
3	6.0127-0.2840i	-6.7474–0.4227i
4	9.3638-0.1898i	-2.6277-6.1027i
5	12.5684-0.1632i	-2.5184-10.6973i

Table 2: Roots of (20) and (21) for $Z = 3.5(1 + i)$, M=0.3 and $k = 7$

2.5 Numerical results

Consider the acoustic propagation in an infinite guide with a monopolar circular source located at the centre of the computational domain for $k = 7$. In the absence of an analytical solution, we compare the numerical solution obtained with the operator DtN to that obtained through the use of layers PML [4] on Σ_- and Σ_+. Figure 2 was made through the code MELINA [3] in the absence of flow and without absorbing while Figure 3 shows the influence of an impedance $Z = 3.5(1 + i)$. In both cases the error is less than 1 %.



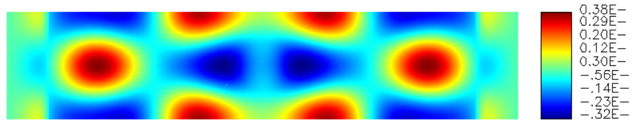


Figure 2: Real part of the acoustic pressure, DtN (top) PML (bottom), M=0, without absorbent, k=7

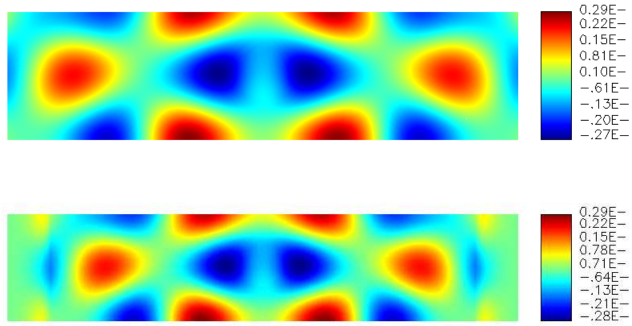


Figure 3: Real part of the pressure, DtN (top) PML (bottom), M=0, $Z = 3.5(1 + i)$, k=7

3 Generalization in 3D: axisymmetric cylindrical duct

We consider here the same problem as before, but axisymmetric. It is a cylinder with a radius R. The problem is set in the $Or\theta z$ plane but by reason of axisymetry it can be restricted to the plane Orz where the z-axis is parallel to the walls of the duct (see Figure (4).

3.1 Hard-walled guide

We have here the same Helmholtz equation and boundary conditions as in the 2D cartesian case, with a xisymmetric boundary condition modeled by a Neumann homogeneous condition:

$$\frac{\partial p}{\partial n} = 0 \quad on \quad \Gamma_0$$

We are looking for solutions by separation of variables: $p\,(r, z) = \vartheta(r) e^{i\beta z}$. This gives us two kinds of modes : $p^{\pm}_\mu\,(r, z) = \vartheta^{\pm}_\mu(r) e^{i\beta^{\pm}_\mu z}$. As in the 2D cartesian case, they are now expressed by:

$$\begin{aligned} \vartheta_0(r) &= \sqrt{2}/R \\ \vartheta_\mu(r) &= \frac{\sqrt{2}}{J_0(k_\mu R) R} J_0(k_\mu r) \quad , \quad \mu \geq 1 \end{aligned} \tag{25}$$

where J_0 is the 0 Bessel order first kind. The dispersion equation is expressed here by:

$$\beta^2_\mu = k^2 - k^2_\mu \tag{26}$$

The DtN operators on Σ_- and Σ_+ are expressed as in the two-dimensional model:

$$T^{\pm}(p) = \mp \sum_{\mu \geq 0} i\beta^{\pm}_\mu (p, \vartheta_\mu)_{\Sigma_\pm} \vartheta_\mu(r) \tag{27}$$

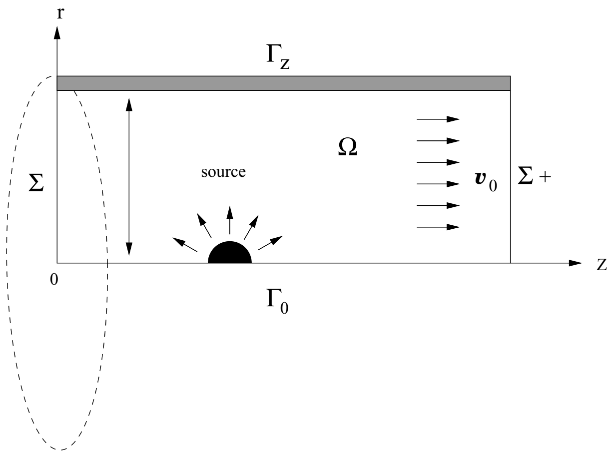


Figure 4: Axisymmetric problem geometry

$$(p, \vartheta_\mu)_{\Sigma\pm} = \iint p\, \bar{\vartheta}\, r\, drdz \tag{28}$$

The variational formultaion is: Find $p \in H^1(\Omega)$ such as $\forall \psi \in H^1(\Omega)$:

$$\iint \nabla p.\nabla\bar{\psi}\, r\, drdz - k^2 \iint p\, \bar{\psi}\, r\, drdz + \int_{\Sigma\pm} T^{\pm}(p)\, \bar{\psi}\, r\, dr = \iint f\, \bar{\psi}\, r\, drdz \tag{29}$$

The difference with the 2D cartesian case is the scalar r that appears in the integrals.

3.2 2D cylindrical lined guide without mean flow

In the presence of absorbent material on the cylinder wall, the boundary conditions change on the wall:

$$\frac{\partial p}{\partial n} = \frac{ikp}{Z} \qquad on \quad \Gamma_Z$$

A search for solutions with separated variables leads to the following equation:

$$-k_\mu \frac{J_1(k_\mu R)}{J_0(k_\mu R)} = \frac{ik}{Z} \tag{30}$$

The normalization of modes becomes:

$$\vartheta_\mu(r) = \frac{1}{R}\sqrt{\frac{2}{J_0^2(k_\mu R) + J_1^2(k_\mu R)}} J_0(k_\mu r), \quad \mu \geq 0 \tag{31}$$

With $J_0^2(k_\mu R) + J_1^2(k_\mu R) \neq 0$, which allows us to determine the critical modes and the critical impedances. One has the same difficulties as in the two-dimensional case requiring the introduction of a new dot product. The transverse waves number k_μ are in table below :

3.3 Lined guide with uniform mean flow

With uniform mean flow ($M = v_0/c$), the Helmholtz equation becomes:

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + (1 - M^2)\frac{\partial^2 p}{\partial z^2} + 2ikM\frac{\partial p}{\partial z} + k^2 p = 0 \tag{32}$$

n	k_μ^+	β_μ^+
1	1.8879 - 0.3454i	6.7501 + 0.0966i
2	4.4605 - 0.4017i	5.4199 + 0.3306i
3	7.3636 - 0.2921i	0.8840 + 2.4329i
4	10.4094 - 0.2159i	0.2917 + 7.7067i
5	10.4094 - 0.2159i	0.2917 + 7.7067i

Table 3: Roots of $-k_\mu J_1(k_\mu R)/J_0(k_\mu R) = ik/Z$ for the corresponding $Z = 4.5(1 + i)$ and β_μ, k=7.

$$\frac{\partial p}{\partial n} = 0 \qquad (\Gamma_0) \tag{33}$$

$$\frac{\partial p}{\partial n} = -\frac{i}{kZ}(M\frac{\partial}{\partial z} - ik)^2 p \qquad (\Gamma_Z) \tag{34}$$

$$\frac{\partial p}{\partial n} = -T_{ZM}^{\pm}(p) \qquad (\Sigma_\pm) \tag{35}$$

Solutions are now expressed by:

$$p_\mu(r, z) = a_\mu J_0(k_\mu r) e^{i\beta_\mu z} \tag{36}$$

with the following conditions:

$$k_\mu^2 = k^2 - (1 - M^2)\beta_\mu^2 - 2kM\beta_\mu \tag{37}$$

$$-k_\mu \frac{J_1(k_\mu R)}{J_0(k_\mu R)} = \frac{i\,(\beta_\mu^2 + k_\mu^2)}{kZ} \tag{38}$$

The new dot product defined in (22) makes it possible to write the DtN operator :

$$T_{ZM}^{\pm}(p) = \pm \sum_{\mu,\nu \geq 0} i\beta_\mu^{\pm}(O^{-1})_{\mu,\nu}\ ((p, \vartheta_\nu))_{\Sigma\pm}\vartheta_\mu(r) \tag{39}$$

with $O_{\mu,\nu} = ((\vartheta_\mu, \vartheta_\nu))$ and

$$((p, \vartheta_\nu)) = \int_{\Sigma\pm} p\, \vartheta\, r\, dr - \frac{i\, M^2}{kZ(1 - M^2)} R\, p(R)\vartheta(R) \tag{40}$$

The scalar r that appears here makes the difference compared to the 2D case.

3.4 Numerical Results

The case of the semi-infinite cylinder is illustrated with a Dirichlet boundary condition on Σ_- and compared with an analytical solution. Then, there is no source $f = 0$. It consists of finding $p \in H^1(\Omega)$ with $p(r, 0) = \vartheta(r)$ such that $\forall \psi \in H^1(\Omega)$, $\psi = 0$ on Σ_-:

$$\iint_\Omega \nabla p.\nabla\psi\, r\, drdz - k^2 \iint_\Omega p\, \psi\, r\, drdz - \frac{ik}{Z}\int_{\Gamma_Z} p\, \psi\, r\, dz + \int_{\Sigma_+} T(p)\, \psi\, r\, dr = 0 \tag{41}$$

Figure (5) presents the results obtained without mean flow and without absorbing while Figure (6) shows the influence of an impedance $Z = 4.5(1 + i)$. In both cases the error is less than 1 %.

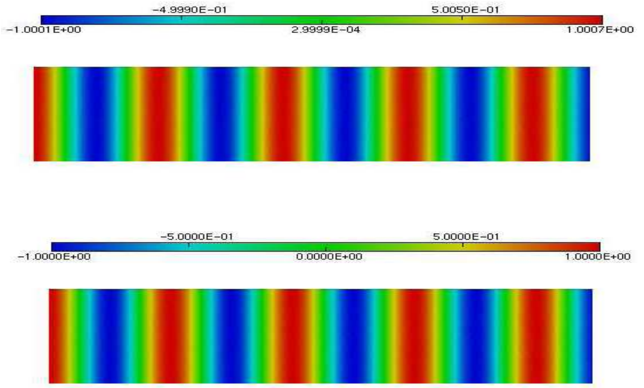


Figure 5: Real part of the acoustic pressure, DtN (on top) analytical Solution (on bottom), M=0, without absorbent, k=7

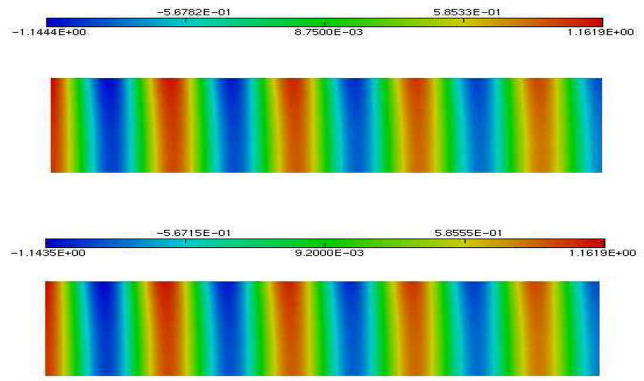


Figure 6: Real part of the pressure, DtN (on top) analytical solution (on bottom), M=0, $\boldsymbol{Z = 4.5(1+i)}$, k=7

4 Conclusions

The numerical determination of the modes in a guide with an absorbing wall in the presence or absence of a uniform flow allowed us to write new transparent boundary conditions . It was shown that the modes form a orthonormed basis within the meaning of a new dot product which we defined in the absence of flow. In the presence of a uniform flow, the modes are not orthogonal any more, but it is still possible to introduce a new dot product including the values of the pressure on the treated wall. The DtN operator can then be calculated with the help of the calculation of a spectral matrix which becomes diagonal when the order of the modes increases.

References

[1] Bonnet-Ben Dhia A. S., Dahi L., Lunéville E., Pagneux V., 2002, Acoustic diffraction by a plate in uniform flow, Math. Models Appli. Sci., 12(5) (2002), 625-647

[2] Bruneau M., 1998, Manuel d'Acoustique Fondamentale, Hermes

[3] Martin M. Librairie Mélina: http://perso.univ-rennes1.fr/daniel.martin/melina/

[4] Bécache E., Bonnet-Ben Dhia A.S. and Legendre G., 2004, Perfectly matched layers for the convected Helmholtz equation, SIAM J. on Numerical Analysis, Volume 42 (1), 409-433

[5] S. Poernomo Sari,E. Redon, J.-F. Mercier, A.-S. Bonnet-Ben Dhia et C. Chambeyron,2007, Condition aux limites transparente pour la propagation acoustique dans un guide recouvert d'un matériau absorbant en présence d'un écoulement uniforme 18ème Congrès Français de Mécanique,Grenoble, 27-31 août 2007

[5] Milton Abramowitz and Irene A. Stegun, Issued June 1964, Handbook of Mathematical Functions, 355-491

[6] Grégoire Derveaux, Décembre 2005, Propagation d'ondes acoustiques dans un guide de section variable. Méthodes des PML, Projet de Calcul Scientifique MA 201

[7] , Catherine Potel, Michel Bruneau, Acoustique générale : Equations différentielles et intégrales, solutions en milieux fluides et solides, applications, Ellipses.