# Vibration of a rib-reinforced floor/ceiling structures with irregularities

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This paper describes mathematical modelling procedure of the rib-reinforced floor/ceiling structures, which are made up of components with irregular shapes and physical parameters. Exact determination of the vibration of a composite structure becomes impossible beyond the low-frequency range because one cannot determine all the necessary parameters of the components. Even if every possible parameter of the structure is known, the results from such deterministic model would not represent the real behaviour of the structure. Therefore, the prediction model in the mid- to high-frequency range must include the effects of the irregularities. We show how the power spectra of the irregular features of each component can be included in the model. The model gives statistical estimates of the solutions, which can give appropriate mean and variance of the vibration of the structure for the given severity of the irregularities.

# 1 Introduction

This paper gives a modelling method for the lightweight rib-reinforced structures, which are often found in floors, ceilings and walls of residential buildings. The focus here is to show how to include the irregular features of the components in the model. Therefore, the model is an extension of the completely deterministic model by the authors [4].

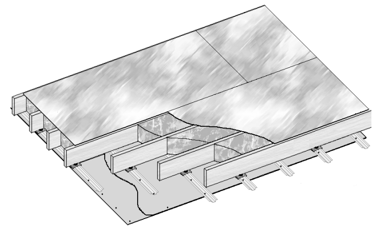

Figure 1: The upper plate (floor) and the ceiling are joined together by the parallel joists. The cavity space is filled with sound absorbing material and the ceiling is connected by the steel battens running perpendicular to the joists.

The modelling method in this paper share the theoretical foundation with the finite element method (FEM). Both methods derive the stiffness matrix from the Lagrangian of the deformation of the structure. At such frequency, FEM is no longer practical due to computational cost. Furthermore, there is no standard procedure in FEM to include the random irregularities in the structures, such as the one in figure 1. In order to deal with the high-frequency vibration, the statistical energy analysis (SEA) has become a popular method in structural vibration in the past 30 years. The SEA is effective in high-frequency vibration, where there are thousands of modes.

There are largely two different modelling methods, SEA and deterministic modelling. We use the deterministic modelling method to study the effects of random irregularities in individual components and junctions. The SEA, on the other hand, deal with a structures as a collection of elements, between which the energy (due to input excitation) flows. The randomness in the structure is included as the fluctuation in the input and coupling between the elements. There are various hybrid methods that combines the SEA and the FEM, for example by Langley, Shorter and Cotoni [14, 16, 15]. Our and their methods both express the irregularity as the addition to the deterministic stiffness matrix.

$$[D_{\text{det}} + D_{\text{ran}}]\, E = f$$

where $D_{\text{det}}$ and $D_{\text{ran}}$ are the deterministic and the random stiffness matrices, and $E$ and $P$ are the energy and the input energy. However, our method is formulated for the deformation of the structure, rather than the energy. Hence, the random part in the components is derived from the Lagrangian of the structure, that is

$$[M_{\text{det}} + M_{\text{ran}}]\, \mathbf{c} = \mathbf{f}$$

where $\mathbf{c}$ is the vector of the coefficients of the Fourier expansion of the deflection of the structure.

A series of papers by Brunskog and Hammer ([1, 2, 3]) shows the effects of cavity space and the joists in the LTFS (lightweight timber-frame structure). The papers by Craik ([5, 6]) show the vibration propagation across junctions between flexible plate and beam. Our extensions over their modells are, first: inclusion of coupling between the plate and the joists (see [4]), and second: inclusion of irregularities of the joist properties. In order to cope with the reallife structures, the model must be able to incorporate the changes without going through the modelling procedure again. For this reason we chose the variational formulation of the system (see [11]). The solution, in this case the deflection of the components, is the minima of the total energy in the structure. We show how to incorporate the interaction conditions between components, using the typical floor/ceiling configuration shown in figure 1.

The idea of using the PSD (power spectral density) of the irregularities can be found in the literature on the scattering by random irregularities (see [9], [10] and [13]). The fluctuation in the scatterer is characterized by the auto-correlation of the fluctuation, which then describes the scattering cross-section. In Howe [9], the PSD of the irregularities in an one-dimensional infinite elastic beam is used to estimate the mean energy transfer through the irregular region. We note that, in Howe [9], the solution (deflection of the beam) is decomposed into *coherent* and *incoherent* parts.

$$B\frac{\partial^4 w}{\partial x^4} + m\left(1 + \xi\left(x\right)\right)\frac{\partial^2 w}{\partial t^2} = 0$$

where $\xi$ is the fluctuation in the mass density. The solution is then split into

$$w = \bar{w} + w^1$$

where $\bar{w}$ and $w^1$ are the coherent and the incoherent solutions, respectively. The averages of the rate of the energy transfer are then expressed using the Fourier transform of the differential equation of the beam. The present paper, on the other hand, the solutions are represented using the Fourier basis functions, which have the equally spaced spatial wave modes. Therefore, the width of the spectra determines the accuracy of the approximation of the solution, rather than the locations of all the four modes of the beam. We use the similar method of calculation when the Fourier transform of the irregularities is incorporated using the convolution between the Fourier transform and the infinite-beam modes.

The small perturbation theory is not appropriate for the class of problem dealing with the interaction between the structure and waves having the similar length scale [12]. The model using the average values of the parameters do not normally give the average solution in the higher frequency range. In other words, such solution do not represent the reality.

## 2 Mathematical modelling

First, we establish the deterministic model. The random irregularities are then incorporated using the Fourier representation of the random components, which may be geometrical shapes or physical parameters, such as stiffness and mass density.

The deflection of the structure is obtained by minimizing the first variation of the Lagrangian of the whole structure. The minimum is computed using the Fourier basis, which expanse the deflection of all individual components. This expansion is made easy because of the rectangular shape of the structure. The Fourier representation of the solution enables us to include the irregularities themselves in the structure as their Fourier representation again.

The deformation of an elastic structure is often analyzed using finite element method (FEM), which produces the stiffness matrix as the result of minimizing the Lagrangian.

$$M\mathbf{c} = \mathbf{f}$$

where $\mathbf{c}$ and $\mathbf{f}$ are the deformation and excitation vectors, respectively. The elements of $M$ are derived from the discretizeation of the integrals of the strain and kinetic energies. In contrast to that, $M$ will be derived here using so-called the global elements, which are the Fourier basis functions. Therefore, the elements of $\mathbf{c}$ will be the coefficients of the Fourier expansion instead of the discretised spacial grid points. The following section will show how the irregularities (deviation from the perfect shape or parameter) can be included in $M$ by simple addition of off-diagonal matrices derived from the Fourier transform of the irregularities.

The Lagrangian for the structure is derived on the assumptions of Euler beam and Kirchhof plate, which have one degree of freedom of movement (vertical). These assumptions are justified for the low- to mid-frequency range, in which the shear deformation is small compare to that of the bending motion. The Lagrangian for deflection $w$ is given by the following formula.

$$\mathcal{L}(w(t)) = \int_0^T [\mathcal{K}(t) + \mathcal{W}(t) - \mathcal{P}(t)]\, dt$$

where $\mathcal{P}$ and $\mathcal{K}$ are the instantaneous potential and kinetic energies of the structure, respectively. $\mathcal{W}$ is the work done on structure by external forces. The problem is further simplified by dealing only with time-harmonic vibration at a single frequency $\omega$. The strain energy and the kinetic energy of a Kirchhoff plate ([8]), which is $A$ meter long in the $x$ axis and $B$ meter wide in the $y$ axis, is given by

$$\int_0^A \int_0^B \left\{ \frac{D}{2} \left[ \left(\nabla^2 w\right)^2 + 2(1-\nu)\left(w_{xx}w_{yy} - w_{xy}^2\right)\right] - \frac{1}{2}\, m\omega^2 w^2 \right\}\, dxdy,$$

and for an Euler beam we have

$$\frac{1}{2}\int_0^A \left\{ EIw_{xx}^2 - m\omega^2 w^2 \right\}\, dx$$

where $w$, $D$, $E$, $I$, $\mu$ and $m$ are vertical deflection, flexural rigidity, Young's modulus, moment of inertia (of the beam), Poisson ratio and mass density, respectively. The true motion of the structure $w(t)$ makes the Lagrangian stationary, that is,

$$\delta\mathcal{L}(\mathbf{w}(t)) = 0,$$

The mounting conditions (or edge conditions) for common floor/ceiling structures are *simply supported*, which simplifies the expansion on to the sine-functions. The deflection of the upper plate ($w_1$), ceiling ($w_3$) and joists ($w_2$) are,

$$w_i(x,y) = \sum_{m,n=1}^{N} c_{mn}^i \phi_m(x)\,\psi_n(y),\ i = 1,3,$$

$$w_2(x,j) = \sum_{m=1}^{N} c_{mj}^2 \phi_m(x),\ \ j = 1,2,...,S_2,$$

where

$$\phi_m(x) = \sqrt{\frac{2}{A}}\sin k_m x,$$

$$\psi_n(y) = \sqrt{\frac{2}{B}}\sin \kappa_n y\ \text{ for } m,n = 1,2,...,N,$$

and $k_m = \pi m/A$, $\kappa_n = \pi n/B$. The number of joists is denoted by $S_2$. Note that the summation of the modes is truncated to $N$ to emphasize the computational aspect of the method. The above basis functions then satisfy the orthogonal relationship.

$$\int_0^A \phi_m\phi_n\, dx = \delta_{mn},\ \int_0^B \psi_m\psi_n\, dy = \delta_{mn}.$$

The acoustic pressure in the cavity is expressed by the Helmholtz equation. Therefore the acoustic pressure can expanded by the Fourier cosine series in the $(x,y)$ plane because the walls of the cavity is assumed to be acoustically hard. By solving the Helmholtz equation using the separation of variables, acoustic pressure $p(x,y,z)$ can be expanded with coefficients $\Gamma_{mn}^{(1)}$ and $\Gamma_{mn}^{(2)}$.

$$p(x,y,z)=\sum_{m,n=0}^{N}\left(\Gamma_{mn}^{(1)}e^{\gamma_{mn}z}+\Gamma_{mn}^{(2)}e^{-\gamma_{mn}z}\right)\alpha_m(x)\,\beta_n(y)$$

where $\gamma_{mn}=\sqrt{k_m^2+\kappa_n^2-k^2}$ and $k=\omega/c$, $c$ being the speed of sound. The modes are

$$\alpha_m(x)=\sqrt{\frac{2}{A}}\cos k_m x,$$

$$\beta_n(y)=\sqrt{\frac{2}{B}}\cos\kappa_n y \quad \text{for } m,n=0,1,...,N.$$

The basis functions for the acoustic pressure again satisfy the orthogonal relationship. Acoustic pressure $p$ is then included using the following coupling conditions

$$\left.\frac{\partial p}{\partial z}\right|_{z=0}=\rho\omega^2 w_1(x,y)\,,\quad \left.\frac{\partial p}{\partial z}\right|_{z=h}=\rho\omega^2 w_3(x,y)\,,$$

where $h$ is the cavity depth. Details of the method of solutions are given in [4, 7]. Figure 2 shows the comparison between the computed and measured average root surface velocity of the ceiling for two different designs.

# 3 Uncertainties in the structure

The uncertainties in the structure will start to have dominating effects in the vibration above the 5th or the 6th resonance frequency as the wavelength of the vibration becomes shorter. Hodges and Woodhouse [12] gave rigorous argument why this should be the case.

The inclusion of random irregularities in the context of the structural vibration are common in the SEA. However, there are many different definitions of randomness and irregularities depending on the structures. The definitions often seem arbitrary. A series of articles by Langley [14, 16, 15] gives detailed description of their definition of the irregularities in the components and the junctions between them. It should be noted that so-called hybrid methods all share the weak-coupling and diffuse-field assumptions from the SEA. These assumptions are likely to be invalid for the lightweight structures because of the strong coupling and highly directional energy propagation ([17] and [18]).

Each irregularity is assumed to be random deviation from the perfect shape or parameter. For example, as given in subsection 3.2, the Young's modulus of the $j$'th joist is given by

$$E+\epsilon(x,j)\,.$$

The deviation function $\epsilon(x,j)$ is derived from the PSD, which may be obtained from the experiments. We also assume that the irregularities are Gaussian random process with zero mean. Therefore, the process is completely determined by the mean and the covariance of the process. This assumption leads to the fact that the realization can be achieved by randomizing the phase, which ensures the correct PSD curve shape.

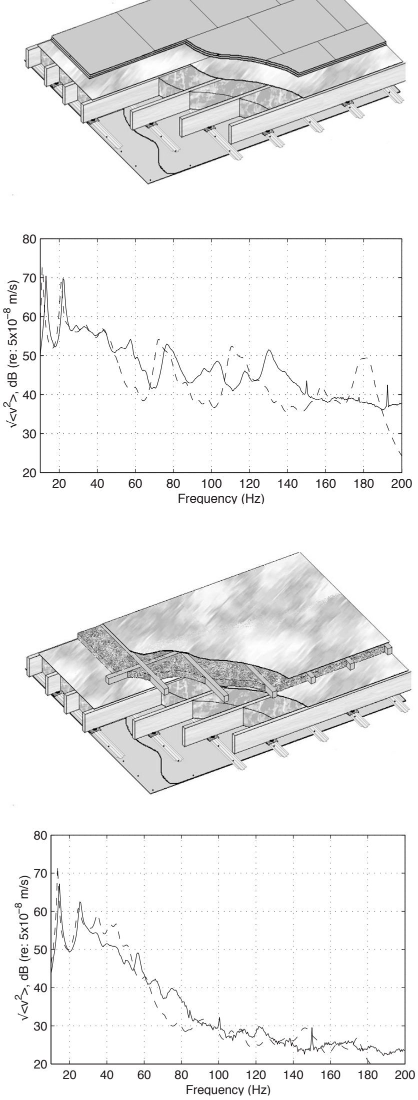


Figure 2: Comparison between the computed and the measured Average square root velocity of the ceiling. Top: more floor boards added to increase the mass and stiffness of the upper layer. Bottom: the floor has additional layer of battens and filling and panels.

As a result of including the irregularities, the Lagrangian of the structure can be represented as

$$\mathcal{L}=\frac{1}{2}\mathbf{c}^{\mathrm{T}}\begin{bmatrix} M^1+J^{11} & J^{12} & \cdots & J^{1L}\\ J^{21} & M^2+J^{22} & & \\ \vdots & & \ddots & \\ J^{L1} & & & M^L+J^{LL}\end{bmatrix}\mathbf{c}+\mathbf{f}^{\mathrm{T}}\mathbf{c}.$$

The stiffness matrix is therefore a random matrix, whose size is determined by the number of Fourier modes and the number of the components. The sub-matrices, which represent the individual components or the coupling condition have the off-diagonal elements with certain random distributions. So far the authors have not been able to find any analytical ways to determine the statistical distribution of the solution from the given distribution of the random stiffness matrix.

## 3.1 Joist shape

As an example of the irregular shape of the joist beams, figure 3 show ten measurements and the PSD of the timber beam shape. The dimension the beams is approximately 0.1m by 0.2m and the length is 2.4m. The PSD in figure 3 indicates that the beams have mostly two or three twists or four at the most.

We note that the joists used in the example LTFS in figure 1 have different size and shape. However, it is reasonable to assume that the PSD of irregular timber beams should have the similarly shaped PSD. A timber beam would have dominating low-frequency deformation and small contribution from higher frequency deformation. The severity of the deformation can be evaluated by the width of the PSD curve. In other words, the narrower the PSD is, the straighter the beam is.

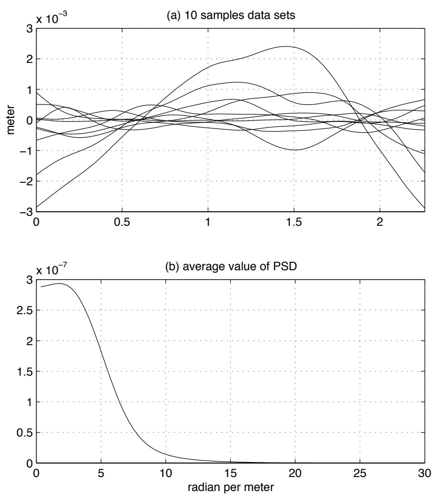


Figure 3: (a) 10 samples of the measurements of the timer shape. (b) average of the power spectra of the timber shape.

Deviation function $\theta$ gives curves instead of straight lines for the contact between the upper plate and the joists. We first take the Taylor expansion of the vibration modes at the contact curves and omit the higher terms because $\theta$ is assumed small. This is a reasonable assumption judging from the data in figure 3.

$$\psi_n(y_j+\theta(x,j)) = \sum_{i=0}^{\infty} \frac{(\kappa_n y_j \theta(x,j))^i}{i!} \frac{d^i\psi_n}{dy^i}(y_j)$$
$$\approx \psi_n(y_j) + \kappa_n^2 y_j \theta(x,j)\,\beta_n(y_j) \quad (1)$$

Note that using only the first term leads to the zero deviation solution.

The above expansion leads to the following modified contact condition.

$$\sum_m c_{mj}^2 \phi_m(x) = \sum_{m,n} c_{mn}^0 \phi_m(x)\,\psi_n(y_j+\theta(x,j))$$

Using the orthogonal relationship and equation 1 gives

$$c_{mj}^2 = \sum_n c_{mn}^1 \psi_n(y_j) + \sum_{m',n} q_{nj} c_{m'n}^1 \int_0^A \theta(x,j)\,\phi_m(x)\,\phi_{m'}(x)\;dx \quad (2)$$

where $q_{nj} = \kappa_n^2 y_j \beta_n$. Hence we rewrite the above equation with matrix notations.

$$\mathbf{c}_2 = \left[J^{12} + L_\theta\right]\mathbf{c}_1.$$

Matrix $L_\theta$ represents the second term on the right hand side of equation 2. The irregularity term is now represented by a simple additional matrix.

The integral in equation 2 can be re-written using the fact that $\{\phi_m\}$ are the Fourier basis functions. The integral is the Fourier transform at the mode numbers $\{k_m, k_{m'}\}$ in a finite interval. Therefore, the elements of matrix $L_\theta$ can be better visualized by rewriting the formula using the convolution of the Fourier transform of $\theta$ and the Delta-function at the mode numbers. More precisely, the elements of $L_\theta$ can be computed by

$$q_{nj}\left[\hat{\theta} * \hat{H} * \{\delta(k_{m-m'}) - \delta(k_{m+m'})\}\right] \quad (3)$$

where $*$ is the convolution. $\hat{H}(\xi)$ is the Fourier transform of rectangular pulse in $[0, A]$, which is required for the finite length of the joists and is given by

$$\hat{H}(\xi) = \frac{\mathrm{i}}{\xi}\left(e^{\mathrm{i}\,A\xi} - 1\right).$$

We used the following integration relationship to obtain the convolution.

$$\int_{-\infty}^{\infty} \theta(x)\,H\left(\frac{x}{A}\right) \sin k_m x \sin k_{m'} x\;dx$$
$$= \int_{-\infty}^{\infty} \theta(x)\,H\left(\frac{x}{A}\right)\left\{e^{\mathrm{i}(k_m-k_{m'})x} - e^{\mathrm{i}(k_m+k_{m'})x}\right\}\;dx$$
$$= \mathcal{F}[\theta H](k_m-k_{m'}) - \mathcal{F}[\theta H](k_m+k_{m'})$$
$$= \hat{\theta} * \hat{H} * (\delta(k_{m-m'}) - \delta(k_{m+m'})).$$

The Fourier transform is denoted by $\mathcal{F}[\cdot]$.

These timber beams are not intended for the floor/ceiling structures because they have many twists and turns.

We rewrite the above formula so that the deviation part can appear as additive terms to the regular term. The potential energy contribution from the beams is then

$$\pi_2 = \frac{1}{2}\mathbf{c}_2^{\mathrm{t}} M_2 \mathbf{c}_2 = \frac{1}{2}\mathbf{c}_1^{\mathrm{t}}\left(J^{12}+L_\theta\right)^{\mathrm{t}} M_2\left(J^{12}+L_\theta\right)\mathbf{c}_1.$$

Higher order terms may be used when more details of the shape deviation have to be included. When only the first order terms are used, the deviation parts become simple additive terms to the deterministic parts.

## 3.2 Young's modulus of the joists

The Young's modulus deviation $\epsilon$ can be incorporate using the similar procedure as before. In this case, the irregularity in Young's modulus need not be small. The strain energy of the joist beam is

$$\frac{I}{2}\int_0^A (\epsilon_0 + \epsilon(x,j))\left\{\frac{d^2 w_2}{dx^2}(x,j)\right\}^2\;dx$$

where $I$ is the moment of inertia.

Using the Fourier expansion of $w_2$ and orthogonality of $\{\phi_m\}$ gives (put the integration formula here)

$$\frac{1}{2}\mathbf{c}_2^{\mathrm{t}}\left\{M_2 + L_\epsilon\right\}\mathbf{c}_2$$

where the elements of $L_\epsilon$ are given by

$$[L_\epsilon]_{m,m'} = Ik_m^2 k_{m'}^2 \int_0^A \epsilon\left(x,j\right)\phi_m\left(x\right)\phi_{m'}\left(x\right)\,dx.$$

Rewriting the integral part of this equation in the similar manner shown in equation 3 gives the $(m,m')$-element of $L_\epsilon$.

$$[L_\epsilon]_{m,m'} = Ik_{mm'}^2\left[\hat{\epsilon}*\hat{H}*\left\{\delta\left(k_{m-m'}\right)-\delta\left(k_{m+m'}\right)\right\}\right],$$

where $\hat{\epsilon}$ is the Fourier transform of $\epsilon$.

# 4 Summary

The past studies by the authors have confirmed that the deterministic model can only predict the vibration up to the first 5 or 6 fundamental modes (around 80Hz). This fact has been known since the 1980's. The SEA can be a powerful tool for the vibration prediction in the range of thousands of modes. This article gives a possible modelling method for the mid-frequency range, which is relevant to the behaviour of the lightweight constructions. The irregularities are included in the stiffness matrix as the off-diagonal elements of each sub-matrix, which represents the interaction between components.

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