Analysis of the resonance frequency shift in cylindrical cavities containing a sphere and its prediction based on the Boltzmann-Ehrenfest principle

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It is known that forces generated by high-level acoustic waves can compensate for the weight of small samples, which can be suspended in a fluid. To achieve this, a standing wave is created in a resonant enclosure, which can be open or closed to the external medium. This phenomenon, called Acoustic levitation, has numerous applications in containerless study and processing of materials. Although it is possible to levitate a sample for long periods of time, instabilities can appear under certain conditions. One of the causes of oscillational instabilities is the change of the resonance frequency of the cavity due to the presence of the levitated object. The Boltzmann-Ehrenfest principle has been used to obtain an analytical expression for the resonance frequency shift in a cylindrical cavity produced by a small sphere, with $kR < 1$, where k is the wavenumber and R is the radius of the sphere. The validity of the Boltzmann-Ehrenfest method has been investigated by means of the Boundary Element Method (BEM) and confirmed with experiments.

1 Introduction

When a single-mode acoustic levitator is driven at a frequency slightly above the resonance of its empty cavity, oscillational instabilities can affect the levitated object. A cause of such instabilities is the change of the resonance frequency of the cavity due to the object [1].

An investigation on the resonance frequency shift of a cylindrical cavity induced by a rigid sphere in its interior is presented in this paper. The position of the sphere can be any point along the axis of the cavity. Several studies considering the same problem are available in the literature. An analytical equation to describe the resonance frequency shift for a plane-wave mode in a chamber containing a rigid sphere was obtained by Leung and co-workers [2]. The calculation of the frequency shift was based on a Green's function, considering the scattering of the acoustic wave on the sphere surface, but neglecting the interactions of the scattered wave on the cavity walls. The equation describes the resonance frequency shift to second order in $k_l R$, where k_l is the wave number corresponding to the resonance of the empty cavity and R is the radius of the sphere. A higher order approximation was, however, not possible.

Numerical calculations of the resonance frequency shift in finite cylindrical ducts produced by blockages of different shapes have been presented by El-Raheb and Wagner [3]. Their calculations are valid for both small and large samples with respect to the dimensions of the duct. The study of the resonance frequency shift in a cylindrical cavity produced by a solid sphere was experimentally extended by Barmatz et al. [4].

An analytical expression for the resonance frequency shift of a plane-wave mode was obtained by studying the local perturbation of the acoustic field around a solid sphere by Curzon and Plant [5]. The expression was valid for very small spheres compared with the wavelength. If the terms depending on $k_l R$ in the equation given by Leung et al. are neglected, it becomes the same as the expression deduced by Curzon and Plant. The equation was also obtained by Rudnick and Barmatz [1], who used a Green's function decomposed into two parts, one corresponding to the contribution of an empty cavity mode whose resonance frequency is close to a shifted resonance of the cavity with the sphere, and the other representing the contribution of the remaining empty cavity modes.

Following a similar procedure as Leung and co-workers, Mehl and Hill obtained theoretically the eigenfrequencies of an arbitrary cavity shape with an internal sphere [6]. In addition, an analytical expression for the resonance frequency shift of any mode in a rectangular cavity containing a small and rigid sphere was deduced by Roumeliotis [7]. For an axial mode, the expression reduces to the same result given by Curzon and Plant.

Recently the resonance frequency shift produced by a solid sphere in an open cavity of a single-axis acoustic levitator has been studied by Xie and Wei [8]. Based on the Boundary Element Method, they gave numerical results, which coincide quite well with Leung's equation for small samples considering an equivalent volume for the cavity.

The resonance frequency shift of a plane-wave mode in a rectangular cavity induced by a spherical solid, with a diameter comparable to the length of the square cross section of the cavity, has been investigated by Cordero and Mujica [9]. Their numerical calculations are in good agreement with experimental results for small spheres compared with the wavelength.

In this paper it is shown that the Boltzmann-Ehrenfest principle can be applied to predict the resonance frequency shift of a cylindrical cavity brought about by a solid sphere. The acoustic field in the empty cavity corresponds to a plane-wave mode. An analytical expression is obtained to describe the resonance shift. It is proved by means of numerical calculations based on the Boundary Element Method (BEM) that this expression gives a better approach than Leung's equation. The BEM is also used to demonstrate the validity of the Boltzmann-Ehrenfest principle. The results are confirmed by experiments.

2 Theory

2.1 The Boltzmann-Ehrenfest principle

According to the Boltzmann-Ehrenfest principle, the ratio between the total energy of the sound field in a resonator (including an object) and the particular resonant frequency of the system is invariant providing the system is friction free and behaves linearly [10,11].

Consider a closed cavity with a stationary sound wave in its interior. The frequency of the acoustic field corresponds to one of the resonances of the cavity. Since the system is assumed to be lossless, the excitation had to cease after the sound field was established.

Let E_0 be the total acoustic energy of the stationary wave in the interior of the empty cavity at the resonance angular frequency ω_0. Now, let a sphere be expanded sufficiently slowly with its center kept fixed in the interior of the cavity. In this process, the volume of the sphere is changed from zero to its final value, V_s. The work done on the system, W, during the total expansion is given by:

$$W = \int_0^{V_s} f_n \, dV \,, \tag{1}$$

where f_n is the time-averaged acoustic normal force per unit area on the surface of the sphere. Thus, the final total acoustic energy in the cavity will be $E_s = E_o + W$. Here the purpose is to find the new resonance angular frequency, ω_s, as a function of the size and position of a rigid sphere along the axis of a cylindrical cavity. According to the Boltzmann-Ehrenfest principle,

$$(\omega_s - \omega_o)/\omega_o = W / E_o \,. \tag{2}$$

2.2 Acoustic force on a rigid sphere

Consider a solid sphere in the interior of a cylindrical cavity with radius R_c and length L. Inside the cavity, with its walls assumed rigid, there is a stationary plane wave in resonance with a wavenumber k_l. Here it is assumed that $k_lR << 1$, where R is the radius of the sphere; therefore, an acoustic resonance in the sphere is not possible. In addition, since the acoustic impedance of the sphere material is much larger than the impedance of the gas, the sphere can be considered rigid.

Assume a cylindrical coordinate system as illustrated in Fig. 1; the position of the center of the sphere is Z. It is considered that the sound pressure amplitude inside the cavity, p, is the sum of the sound pressure amplitude in a plane-wave mode of the empty cavity, $p_l = A\cos(k_l z)$, plus the sound pressure amplitude of the scattered acoustic wave, p_s. By using spherical coordinates with the origin at the center of the sphere, p_l can be expressed as

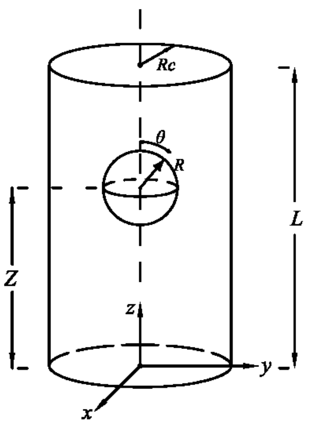

Fig. 1. Cylindrical cavity with the sphere in its axis and the reference system used.

$$\begin{aligned} p_l &= A\cos[k_l(r\cos\theta + Z)] \\ &= \frac{A}{2}\left(e^{ik_lZ}e^{ik_lr\cos\theta} + e^{-ik_lZ}e^{-ik_lr\cos\theta}\right). \end{aligned} \tag{3}$$

Now, with the identity [12]

$$e^{ik_lr\cos\theta} = \sum_{n=0}^{\infty}(2n+1)i^n j_n(k_lr)P_n(\cos\theta)\,, \tag{4}$$

where $j_n(k_lr)$ is the spherical Bessel function and $P_n(\cos\theta)$ is the Legendre Polynomial, Eq. (3) yields

$$p_l = \sum_{n=0}^{\infty} B_n(2n+1)j_n(k_lr)P_n(\cos\theta)\,, \tag{5}$$

where

$$B_n = (A/2)\,i^n\left[\exp(ik_lZ) + (-1)^n\exp(-ik_lZ)\right]. \tag{6}$$

It is assumed that the sphere is small enough and that it is sufficiently far from the cavity walls that the scattered wave is due only to the presence of the sphere. In this way, let the complex amplitude of the scattered acoustic wave be described by [2,13]

$$p_s = \sum_{n=0}^{\infty} C_n h_n^{(1)}(k_lr)P_n(\cos\theta)\,, \tag{7}$$

where $h_n^{(1)}(k_l r)$ is the spherical Hankel function of the first kind; C_n is a constant determined from the condition that the component of the particle velocity normal to the surface of the sphere must be zero (rigid sphere), which gives

$$C_n = -B_n\frac{(2n+1)j_n'(k_lR)}{h_n^{(1)'}(k_lR)}\,. \tag{8}$$

As a result, the total sound pressure amplitude at a point on the sphere surface is given by

$$p(R,\theta) = \sum_{n=0}^{\infty}(2n+1)B_nP_n(\cos\theta)\left[j_n(k_lR) - \frac{j_n'(k_lR)}{h_n^{(1)'}(k_lR)}h_n^{(1)}(k_lR)\right]. \tag{9}$$

In the limit of small k_lR, Eq. (9) can be approximated at the order of $(k_lR)^2$; in this way, the first three terms of the summation are conserved. In addition, by substituting the spherical Bessel and Hankel functions in terms of sine and cosine functions, and by expressing these trigonometric relations as series of k_lR, after lengthy but straightforward calculations, Eq. (9) reduces to

$$\begin{aligned} p(R,\theta) &= A\cos(k_lZ)\left[1 - \frac{1}{2}(k_lR)^2\right] \\ &- A\frac{3}{2}k_lR\sin(k_lZ)P_1(\cos\theta) \\ &- A\frac{5}{9}(k_lR)^2\cos(k_lZ)P_2(\cos\theta)\,. \end{aligned} \tag{10}$$

An inviscid fluid and an immovable sphere have been considered; moreover, taking into account that the system is independent of the azimuthal angle, it follows that the fluid on the surface of the solid will oscillate only in the θ direction. Thus, the component in that direction of the complex particle velocity amplitude is determined by means of the linear Euler's Equation as

$$u_\theta = (i\omega\rho r)^{-1}(\partial p/\partial\theta)\,. \tag{11}$$

The time-averaged acoustic normal force per unit area on the sphere is given by [13]

$$f_n = \frac{1}{2\rho c^2}\langle p^2\rangle - \frac{1}{2}\rho\langle \boldsymbol{u}\bullet\boldsymbol{u}\rangle\,, \tag{12}$$

where $\langle\cdot\rangle$ indicates the time average; p and $\boldsymbol{u}$ are, respectively, the sound pressure and the particle velocity evaluated on the surface of the sphere.

The total normal acoustic force exerted on the surface of the sphere, F_n, can be now evaluated as

$$F_n = \oint_S f_n(R,\theta)\,da\,, \tag{13}$$

where S is the total surface of the sphere.

Use of Eqs. (10) and (11) in Eq. (12) to substitute the resulting expression for f_n in Eq. (13) gives

$$F_n = \frac{A^2 \pi R^2}{2\rho c^2} \left\{ -\frac{1}{2} + \frac{5}{2}\cos(2k_l Z) + \left[-\frac{7}{108} - \frac{169}{108}\cos(2k_l Z) \right] (k_l R)^2 + \left[\frac{1}{9} + \frac{1}{9}\cos(2k_l Z) \right] (k_l R)^4 \right\} . \quad (14)$$

2.3 Resonance frequency shift

Substitution of Eq. (14) into Eq. (1) produces

$$W = \frac{A^2}{4\rho c^2} V_s \left\{ \left[-\frac{1}{4} + \frac{5}{4}\cos(2k_l Z) \right] + \left[-\frac{7}{360} - \frac{169}{360}\cos(2k_l Z) \right] (k_l R)^2 + \left[\frac{1}{42} + \frac{1}{42}\cos(2k_l Z) \right] (k_l R)^4 \right\} . \quad (15)$$

The total acoustic energy for a plane-wave mode in the cylindrical cavity without the sphere is $E_o = V_c A^2/(4\rho c^2)$, where $p(z,t) = A\cos(k_l z)\exp(-i\omega t)$ has been used and V_c is the volume of the cavity. Note that $k_l = l\pi/L$, where l is a natural number. Finally, the use of this result and Eq. (15) allow the calculation of the resonance frequency shift for the cylindrical cavity produced by a small solid sphere in its interior according to Eq. (2):

$$\frac{\omega_s - \omega_l}{\omega_l} = \frac{V_s}{V_c} \left\{ \left[-\frac{1}{4} + \frac{5}{4}\cos(2k_l Z) \right] - \left[\frac{7}{360} + \frac{169}{360}\cos(2k_l Z) \right] (k_l R)^2 + \left[\frac{1}{42} + \frac{1}{42}\cos(2k_l Z) \right] (k_l R)^4 \right\} , \quad (16)$$

where ω_l is the angular frequency of the plane-wave mode l in the empty cylindrical cavity.

3 Experiment

The measurements of the resonance frequency shift were carried out using the experimental setup illustrated in Fig. 2. The cylindrical cavity was formed using an acrylic tube with an inner radius of 2.54 cm, a length L = 10.57 cm, and a wall 0.6 cm thick. The acoustic wave was produced by means of a driver from a horn loudspeaker, which was coupled to the cylindrical cavity by means of a wave guide made of PVC. The opening of the wave guide into the cavity had an inner diameter of 0.3 cm and was centered on the axis of the cylinder. The top of the cavity was also made of PVC, and it had a thickness of 1.0 cm. Several spherical samples, made of glass, were used. They were suspended along the axis of the cylinder by means of a thin optic fiber 0.25 mm in diameter attached to them.

The sound pressure amplitude was measured on the top of the cavity by using a probe microphone (B&K 4182), which was located 1.25 cm from the axis of the cylinder. The resonance frequency of the cavity was determined from the curve of the frequency response obtained with a lock in amplifier (SR 8500). In this case, a sinusoidal sweep signal generated by the lock in amplifier was used as the signal to excite the driver.

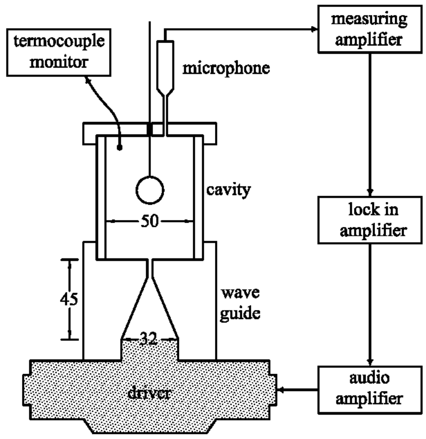

Fig. 2. Diagram of the experimental setup used. The indicated dimensions are in mm.

The main source of error in the experiment was the variation of the temperature. For that reason, the temperature was registered with a thermocouple, type E, connected to a thermocouple monitor (SR 360).

To reduce the heat delivered by the driver, the amplitude of the voltage to excite that transducer was required to have a relatively low value. The floor noise in the empty cavity was first measured, which was less than 25 dB in all cases. The voltage was adjusted to produce about 95 dB at resonance at the top of the empty cavity. For this value, the heat delivered by the voice-coil of the driver increased the temperature inside the cavity sufficiently slow, and the effect of the floor noise was negligible.

Measurements were made of the resonance frequency as a function of the position of a given sphere along the axis of the cavity. In each set of measurements, the initial position was close to the opening of the wave guide into the cavity; the position of the sphere was changed until it was close to the top of the cavity, and the measurements were repeated from the top to the initial position. In this way two measurements were made at each position, and the mean value is reported. The resonance frequency of the empty cavity was measured at the beginning and at the end of each set of measurements of the resonance frequency shift. The mean value of these two measurements was considered as the resonance frequency of the empty cavity.

4 Numerical simulations

The resonance frequencies for the various configurations were calculated by using the Boundary Element Method (BEM). A formulation for axisymmetrical bodies was used, where only the generator of the structure is meshed into one-dimensional elements [14], as shown in Fig. 3.

As a rule of thumb, it is required that a BEM mesh contains at least six elements per wavelength. In the simulations for this paper, 24 quadratic elements were used along the generator of the cylindrical cavity, distributed with 20 elements on the side and 7 on each lid. The spherical object was modeled using 10 elements in all cases. As a check, some resonance frequency calculations were repeated with rougher meshes, yielding negligible differences. The results were further verified using the analytical solution for the eigenfrequencies of an empty cylindrical cavity. The numerically calculated eigenfrequency never fell more than 1 Hz apart from the theoretical value.

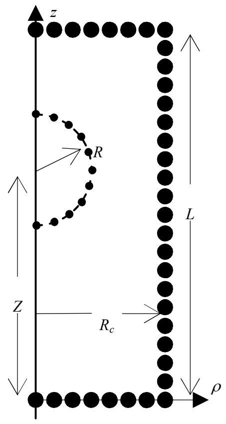

Fig. 3. Example of the axisymmetrical BEM mesh used in the calculations.

The BEM does not readily provide direct computation of the eigenfrequencies; it shows the eigenfrequencies as maxima, as a function of the frequency, of the condition numbers of its coefficient matrices. In the present calculation, an iterative process has been employed to find these maxima, with successive BEM calculations at varying frequency. If this iteration process is properly adjusted, the BEM gives eigenfrequency values as accurate as the Finite Element Method (FEM) can. There were two reasons for using BEM instead of FEM in this investigation: i) the geometrical modifications for moving or scaling the small sphere are simple to do automatically with the discretized BEM model, whereas a FEM model would require the geometry to be re-meshed for each configuration, and ii) the BEM code was already developed for research purposes by a group that includes one of the authors.

In this paper, the eigenfrequencies of the cavity-sphere setup are calculated as a function of the sphere position, and the sphere radius. In the first case, calculations were made in steps of 0.005/L. In the second case, the sphere radius was increased in steps of 0.02 cm. It took about three minutes per eigenfrequency in a modern desktop computer.

5 Results and analysis

An example of the resonance frequency shift as a function of the position of the sphere is shown in Fig. 4, where the studied mode corresponded to l = 3, and the experimental resonance frequency of the empty cavity was 4838.1 Hz. A sphere with a diameter of 2.15 cm was used, which gave a value of k_3R = 0.96. A very good agreement between the experimental data and the results of the numerical simulations can be observed. The contribution of the terms with the factor $(k_lR)^4$ in Eq. (16) is relatively small; therefore, the corresponding curve is an approximation to second order in k_lR. This curve compares very well with the experimental data around the maxima of the relative resonance frequency shift. However, the difference between this curve and the measurements increases in the neighborhood of the minima. It can also be observed that Eq. (16) gives a better approximation to the experimental results than Leung's equation.

The theoretical process for the frequency displacement based on the Boltzmann-Ehrenfest principle was reproduced using the Boundary Element Method. The process is as follows: 1) The sound pressure is solved for a cavity with an object. The excitation should produce a standing wave of amplitude A, which should be the same assumed for the analytical estimation of the total energy in the empty cavity. The cavity walls and the object are defined as rigid and the BEM is used to obtain the sound pressure on them. This calculation is repeated for different radii in steps of 0.2 mm. 2) For each radius, the sound pressure is obtained on discrete points equally spaced along the generator of the sphere. Euler's equation is applied using a discrete gradient to obtain the particle velocity. 3) The time-averaged pressure exerted by the sound field on the sphere surface is obtained by applying Eq. (12) with the calculated values of sound pressure and particle velocity. 4) The total normal force exerted on the sphere is calculated using a discrete integral. 5) The work W for a specific radius is obtained by integrating numerically the force as in (15). Finally, 6) The frequency displacement for every radius is then calculated using eq. (2).

This procedure provides an alternative derivation of the frequency displacement, also based on the B-E principle, but through a numerical calculation. The result is closer to real conditions than Eq. (16) because the approximation k_lR << 1 is not considered and the interaction between the object and the cavity walls is taken into account. This calculation serves as a further verification of the Boltzmann-Ehrenfest principle.

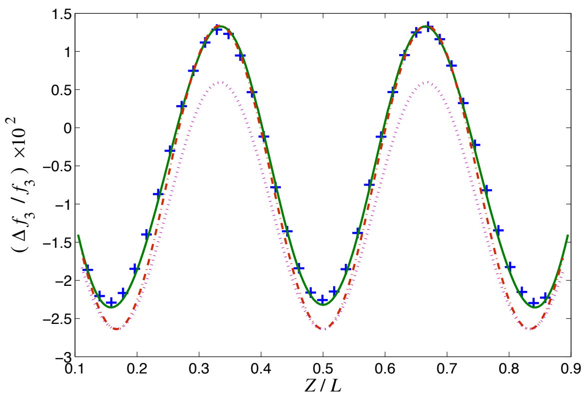

Fig. 4. Resonance frequency shift for the third mode of a cylindrical cavity as a function of the position of a sphere along the axis: Experiment (+), numerical simulations (—), results based on the Boltzmann-Ehrenfest principle (· − ·), and predictions with Leung's equation (····).

The effect of the size of the sphere on the resonance frequency shift was also analyzed. The result for the mode l = 3 with a sphere centered in the cylindrical cavity can be seen in Fig. 5. The numerical calculation based on the Boltzmann-Ehrenfest principle agrees very well with the numerical simulations obtained by means of the BEM; therefore, it confirms the validity of the Boltzmann-Ehrenfest principle. Eq. (16), which gives the same result as Leung's equation when the sphere is located in a pressure node of a pure plane-wave mode, gives a very good result for values of k_3R <0.7.

Graphs for the center of the sphere located at the position corresponding to a pressure antinode of a plane-wave of the empty cavity are shown in Fig. 6. Here positive resonance frequency shifts are produced. The results for the mode l = 3 and Z = L/3 are plotted as a function of the size of the sphere. It can be observed that in this case Eq. (16) gives a better approximation than Leung's equation. In addition, the numerical calculation based on the Boltzmann-Ehrenfest principle gives a very precise result.

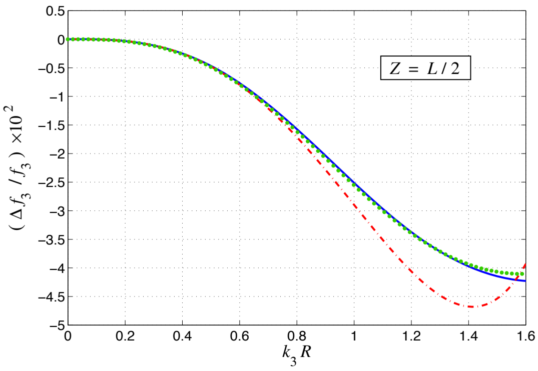

Fig. 5. Dependence of the resonance frequency shift on the radius of the sphere for the third plane-wave mode. The object is located in the center of the cavity. Numerical simulations based on the BEM (—) and numerical approximation of the Boltzmann-Ehrenfest principle (· · ·). For this position of the sphere, Eq. (16) gives the same curve (· – ·) as Leung's equation.

6 Conclusion

The Boltzmann-Ehrenfest principle allowed the deduction of an analytical expression to predict the resonance frequency shift of a plane-wave mode in a cylindrical cavity caused by a solid sphere located at any position along the axis of the cavity. This method is more direct and simpler than a previously reported one based on a Green's function. In contrast with the latter, it was possible predict the resonance frequency shift to the fourth order in $k_l R$. Furthermore, the obtained expression was shown to be more general than the one previously published; the equation based on the Boltzmann-Ehrenfest principle agreed better with experimental and numerical results for large spheres. By means of a numerical implementation, the validity of this principle was also confirmed.

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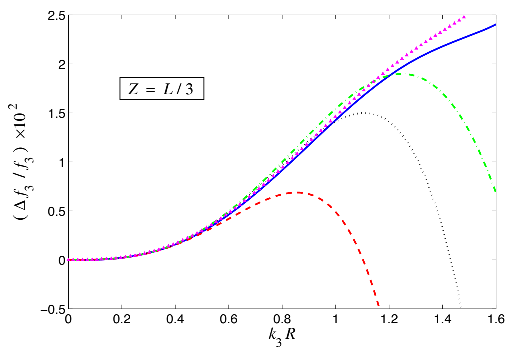

Fig. 6. Resonance frequency shift of the $l = 3$ mode as a function of the size of a sphere, whose center is at $z = L/3$. Numerical simulations based on the BEM (—), Leung's equation (– –), results based on the Boltzmann-Ehrenfest principle [Eq. (16)] approximated to $(k_3R)^2$ (····) and to $(k_3R)^4$ (· – ·), and numerical approximation of that principle (▲)