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Optimal array pattern synthesis with desired magnitude response

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Letting Euclidean norm be the performance parameter, the task of finding the best approximation of a complex function in a finite dimension subspace leads to a convex optimization problem that can be easily solved by the least-squares method. However, this procedure leads to a sub-optimal solution in applications that have no phase requirements on the approximated function. In this case, semidefinite programming has been used to obtain optimal magnitude responses. In this work, this non-convex optimization problem is dealt with by using an iterative method based on the least-squares, which is illustrated on directivity synthesis by spherical loudspeaker arrays. Usually, instead of synthesize directly the desired pattern, the strategy adopted is to reproduce its truncated spherical harmonic representation. The truncation order is determined by the number of drivers of the spherical array. It is shown that truncation error and signal powers can be significantly reduced if phase error is neglected, providing potential means to improve directivity synthesis for applications requiring only magnitude response.

1 Introduction

Letting Euclidean norm be the performance parameter, the task of finding the best approximation of a complex function in a finite dimension subspace leads to a convex optimization problem that can be easily solved by the least-squares method. However, this procedure leads to a sub-optimal solution in applications that have no phase requirements on the approximated function. This non-convex problem arises in array pattern synthesis with desired magnitude response (phase not concerned).

A semidefinite relaxation is used in [1], so that such a non-convex problem is approximated by a convex one. In the present work, a simple iterative method based on the least-squares is used, which presents good results if the algorithm is properly initialized. The method is illustrated on directivity synthesis by spherical loudspeaker arrays.

Spherical harmonic functions constitute a natural basis for representation of compact sound source directivities, since they emerge from the solution of the Helmholtz equation in spherical coordinates. Therefore, the control strategy generally adopted is to provide the spherical array with some preprogrammed basic directivities corresponding to spatial harmonic patterns, cf. [2, 3, 4]. Then, different directivities can be achieved simply by changing the gains associated with the basic directivities, so that it is not necessary to redesign the filters when a different target pattern is desired. In order to calculate these gains, the spherical harmonic decomposition (SHD) of the target pattern must be carried out. It is important that synthesized basic directivities present magnitude and phase accuracy. However, if the desired pattern is frequency dependent, its SHD leads to coefficients that vary with frequency, so that additional filtering is necessary. In this case, spherical harmonic representation is still useful, as directivity pattern can be rotated simply by changing the gains.

The SHD of the target pattern must be truncated since spherical arrays have a limited number of drivers, e.g., an icosahedral source with 20 drivers can reproduce spherical harmonics up to degree 3 [4]. Generally, high degree harmonics are necessary to describe a sound field, especially at high frequencies when sound sources tend to be more directional. Even though a spherical source with 120 drivers has already been built (see [5]), to increase the number of drivers is not a cost-effective solution. In this work, the improvements in the directivity synthesis that can be achieved by ignoring the phase of the target pattern are discussed.

2 Formulation and algorithm

Let $\mathsf{b} \in \mathbb{C}^s$ contain s samples of a target function, $\mathsf{A} \in \mathbb{C}^{s\,x\,L}$ be a basis with L elements and $\mathsf{x} \in \mathbb{C}^L$ contain coefficients that express b in the subspace spanned by A. The following convex optimization problem can be formulated to obtain x (least-squares):

(I) $$\min_{x} \| \mathsf{Ax} - \mathsf{b} \|_2$$

To solve problem (I), an inner product must be defined.

The functions considered in the present work are defined on a spherical surface. These functions are sampled so that $\theta_t = t\Delta\theta$ and $\varphi_m = m\Delta\varphi$; where θ is the elevation angle, φ is the azimuth angle, t = 0, 1, 2 … T-1 and m = 0, 1, 2 … M-1. Then, s = TM is the number of samples, $\Delta\theta = 180^0/(T-1)$ and $\Delta\varphi = 360^0/M$. The meshed spherical surface motivates the use of the following inner product:

$$\langle \mathsf{u},\mathsf{v} \rangle = \mathsf{v}^H \mathsf{W} \mathsf{u} \quad (1)$$

where u, $\mathsf{v} \in \mathbb{C}^s$ and $\mathsf{W} \in \mathbb{R}_+^{S\,x\,S}$. W is diagonal and it contains non-dimensional area weight factors that are determined by surface integration over appropriate sections of the sphere. Thus, the diagonal terms of W, w_i, are given by

$$w_i = \begin{cases} \dfrac{1}{4\pi} \displaystyle\int_{\varphi_m-\Delta\varphi/2}^{\varphi_m+\Delta\varphi/2} \int_{0}^{\Delta\theta/2} \sin(\theta) d\theta\, d\varphi = \frac{1}{M}\sin^2\left(\frac{\Delta\theta}{4}\right), & \text{if } t = 0 \text{ or } T-1 \\ \dfrac{1}{4\pi} \displaystyle\int_{\varphi_m-\Delta\varphi/2}^{\varphi_m+\Delta\varphi/2} \int_{\theta_t-\Delta\theta/2}^{\theta_t+\Delta\theta/2} \sin(\theta) d\theta\, d\varphi = \frac{1}{M}\sin(\theta_t)\sin\left(\frac{\Delta\theta}{2}\right), & \text{if } 1 \le t \le T-2 \end{cases} \quad (2)$$

where i = mT + t + 1.

Thus, the solution of problem (I) is given by [6]

$$\mathsf{x}^* = (\mathsf{A}^H \mathsf{W} \mathsf{A})^{-1} \mathsf{A}^H \mathsf{W} \mathsf{b} \quad (3)$$

Resulting x^* treats magnitude error and phase error equally. However, if phase error is ignored, problem (I) must be replaced by problem (II), which is not convex.

(II) $$\min_{x} \| |\mathsf{Ax}| - |\mathsf{b}| \|_2$$

Now, let $\mathsf{B} \in \mathbb{R}_+^{S\,x\,S}$ be a diagonal matrix so that $\mathsf{B} = \mathrm{diag}(|\mathsf{b}|)$ and $\mathsf{y} \in \mathbb{C}^s$ contain only phase information, so that $(\mathrm{diag}(\mathsf{y}))^H \mathrm{diag}(\mathsf{y}) = \mathsf{I}$. Then, problem (II) can be reformulated as [1]

(III) $$\begin{aligned} &\underset{x,y}{\text{minimize}} \; \| \mathsf{Ax} - \mathsf{By} \|_2 \\ &\text{subject to} \; (\mathrm{diag}(\mathsf{y}))^H (\mathrm{diag}(\mathsf{y})) = \mathsf{I} \end{aligned}$$

Problem (III) is not convex in optimization variables x and y. However, when y is fixed, it is convex in x and the optimal solution is analogous to that one expressed in Eq.(3):

$$x^* = (A^H W A)^{-1} A^H W B y \quad (4)$$

This suggests a method to solve (III) in which y is iteratively updated according to $y_i = e^{j \arg(A_i x)}$, where $j = \sqrt{-1}$, A_i is the i-th row of A and y_i is the i-th component of y. The algorithm is summarized below.

1) Initialize y

2) Repeat until $\frac{[[y_i]-[y_{i-1}]]^H[[y_i]-[y_{i-1}]]}{[y_i]^H[y_i]} < \varepsilon$

2.1) $x = (A^H W A)^{-1} A^H W B y$,

2.2) $y_i = e^{j \arg(A_i x)}$; $y = [y_i]$

3) Optimal solution $x^* = x$, $y^* = y$

In [1], a different method is used to solve problem (III), which is based on a semidefinite relaxation, i.e., this non-convex optimization problem is approximated by a convex one. The iterative method used in this work does not ensure optimality of the solution, but it performs better than the standard least-squares and it provides good results if the algorithm is properly initialised. Its main advantage over the semidefinite approach is that it is a simple and easy-to-implement method.

3 Directivity synthesis by spherical loudspeaker arrays

In this work, the spherical harmonic strategy is adopted to synthesize the target directivity by a spherical loudspeaker array, i.e., the spherical source is provided with preprogrammed basic directivities corresponding to spatial harmonic functions and the target pattern must be decomposed in this basic functions. Thus, the difference between the target pattern and the synthesized one is due to the target decomposition and to the error involved in the synthesis of the basic directivities (spherical harmonics).

3.1 Synthesis of spherical harmonics

Let $A \in \mathbb{C}^{s\,x\,L}$ have the directivities (magnitude and phase) of the L drivers of the loudspeaker array as columns, $x \in \mathbb{C}^L$ contain driver velocities, $S \in \mathbb{C}^{s\,x\,(N+1)^2}$ contain samples of spherical harmonic functions up to degree N as columns, $c \in \mathbb{C}^{(N+1)^2}$ have complex coefficients and $b = Sc$. Then, the optimum driver velocities are given by eq.(3).

In this work, A is obtained by using an analytical model for the sound radiation of the spherical array that takes into account the frequency dependence of the loudspeaker directivities. This model consists in a set of drivers mounted on a spherical surface. Each driver is modeled as a convex spherical cap that vibrates with a constant radial velocity along its surface, as described in [4].

Let c_j be the j-th element of c, δ be the Kronecker delta and S_i be the i-th column of S. By doing $c_j = \delta_{ji}$, Eq.(3) yields to optimum cap velocities (in the least-squares sense) to reproduce the spherical harmonic function given in S_i. If this procedure is repeated from i = 1 up to $i = (N+1)^2$, a matrix $X^* \in \mathbb{C}^{L\,x\,(N+1)^2}$ containing optimal solutions for each spherical harmonic as columns is obtained. Since X^* is evaluated, optimal solution for any linear combination of spherical harmonics up to degree N is given by X^*c.

It is known that the function spaces spanned by spherical harmonics of the same degree are linear subspaces that are invariant with respect to rigid rotation through spatial angles θ and φ [7]. For example, if a given pattern is in the subspace generated only by harmonics of degree 3, any rotation of this pattern also possesses a spherical harmonic expansion consisting only of harmonics of degree 3. Then, if $b_n \in \mathbb{C}^s$ contains samples of a function in the subspace generated by spherical harmonics of degree n, it can be expressed as $b_n = S_n c_n$, where $c_n \in \mathbb{C}^{2n+1}$ and $S_n \in \mathbb{C}^{s\,x\,2n+1}$ is a matrix whose columns contain spherical harmonics of degree n and orders from –n to n.

Now, let $c_n^H c_n = 1$ and $X_n^* \in \mathbb{C}^{L\,x\,2n+1}$ have the optimal cap velocities associated with the 2n+1 columns of S_n. Then, maximum and minimum singular values of $W^{1/2}(AX_n^* - S_n)$ provide, respectively, upper and lower error bounds associated with the subspace spanned by spherical harmonics of degree n. The directivity patterns associated with such bounds can be determined by examining the right-singular vectors obtained in the singular value decomposition [3].

3.2 Spherical harmonic decomposition

Let $p \in \mathbb{C}^s$ represent a given sound pressure field. Problem (I) can be solved in order to obtain c that better express p in the subspace defined by S according to the Euclidean norm. This process is called spherical harmonic decomposition (SHD). N is the truncation number of the SHD. Then,

$$\text{(IV)} \qquad \min_{c} \| Sc - p \|_2$$

The optimal solution is given by [6]

$$c^* = (S^H W S)^{-1} S^H W p \quad (5)$$

Resulting c^* treats magnitude truncation error and phase truncation error equally. Normalized magnitude error of the SHD can be evaluated by Eq.(6).

$$E_{SHD} = \frac{\| |Sc^*| - |p| \|_2}{\| |p| \|_2} \quad (6)$$

However, since E_{SHD} is not minimized by solving problem (IV), it is expected to achieve a smaller magnitude error in the SHD if phase error is ignored. This can be dealt with by replacing problem (IV) by problem (V), which can be solved iteratively as described in session 2.

$$\text{(V)} \qquad \begin{aligned} &\underset{c,u}{\text{minimize}} \| Sc - Pu \|_2 \\ &\text{subject to} \left(\text{diag}(u)\right)^H \left(\text{diag}(u)\right) = I \end{aligned}$$

In problem (V), $\mathsf{P} \in \mathbb{R}_+^{s \times s}$ is a diagonal matrix so that $\mathsf{P} = \mathrm{diag}(|\mathsf{p}|)$ and $\mathsf{u} \in \mathbb{C}^s$ contain only phase information, so that $(\mathrm{diag}(\mathsf{u}))^H \mathrm{diag}(\mathsf{u}) = \mathsf{I}$.

Normalized magnitude error due to the truncated SHD can be evaluated by Eq.(6). The final magnitude error is given by Eq.(7), which is a measure of the distance between the target sound field and the synthesized one.

$$E = \frac{\left\| \left| \mathsf{X}^{*} \mathsf{c}^{*} \right| - \left| \mathsf{p} \right| \right\|_2}{\left\| \left| \mathsf{p} \right| \right\|_2} \qquad (7)$$

4 Results

Here, some simulation results are presented in order to illustrate the improvements that can be achieved by ignoring phase error in the SHD.

The following values were used in the simulations: L = 20 (icosahedral source), s = 79x40 = 3160 points, c_0 = 343m/s, ρ_0 = 1.21kg/m^3 and r/a = 20; where c_0 is the sound speed, ρ_0 is the equilibrium density of the medium, r is the radius of the sphere on which the sound pressure is evaluated and a is the radius of the spherical array. All drivers of the source have the same size, which was chosen to be as large as possible, so that overlap does not occur. The directivity of each driver of the array was evaluated analytically, according to the multipole source model presented in [4]. The drivers were supposed to have the same characteristics.

The directivity of a single spherical cap of the icosahedral source was used as target pattern. It was calculated by using a spherical harmonic expansion up to degree 9 and the cap velocity was made frequency dependent in order to provide a constant sound pressure level of 80dB (ref. 20μPa) in the main radiation direction. So, the presented results show the ability of several drivers in reproducing a single driver. There would be no approximation error if driver directivities were used as a basis for directivity representation. However, since SHD of the target function is performed, large errors arise in the final pattern that is synthesized by the spherical array, as it will be shown.

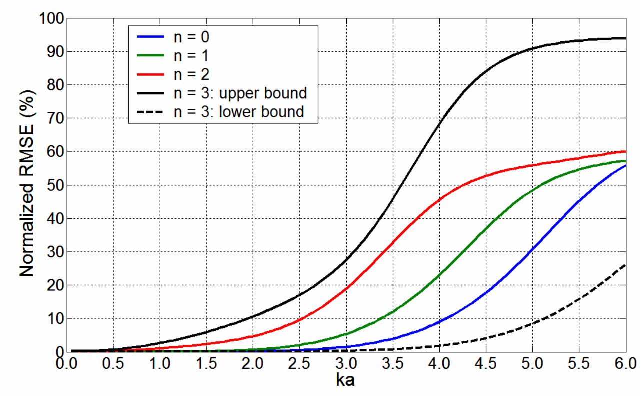

Fig.1 Normalized RMSE arisen in reproducing functions in the subspace spanned by spherical harmonics of degree n by an icosahedral source.

Figure 1 shows the root mean square errors (RMSE) that arise in reproducing functions in spherical harmonic subspaces up to degree 3 by an icosahedral source. Theses curves were obtained as described in the end of session 3.1. Only one curve was plotted for n = 0, 1 and 2 for clarity, since computations have shown that upper and lower bounds for each one of these subspaces are not distinguishable. It can be verified that synthesized basic patterns become less accurate as ka and n increase (k is the wave number).

The magnitude errors - calculated by Eq.(6) - due to the SHD of the target pattern are shown in Fig.2 for different truncation numbers (N). The error decreases as N increases, as expected. It is shown that magnitude error can be drastically reduced if the phase of the target pattern is ignored. A good spherical harmonic representation of the target magnitude can be achieved by using N = 1.

The iterative method was initialized by letting u contain the phase of the target pattern (refer to problem (V)). The normalized mean square difference between two consecutive u was used as convergence criteria (ε=0.03%).

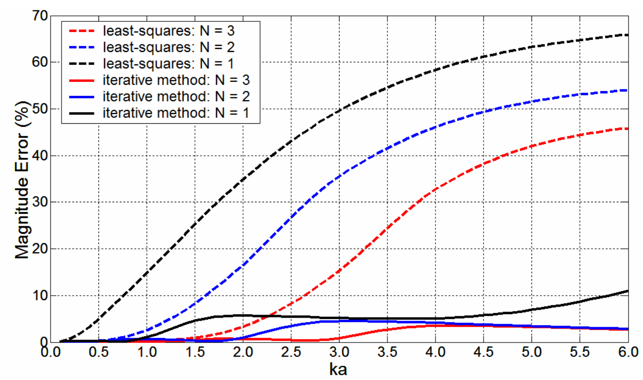

Fig.2 SHD magnitude errors – different truncation numbers and optimization procedures.

Figure 3 compares the harmonic coefficients obtained by solving problem (IV) and problem (V) for N = 3. For each spherical harmonic subspace, the curves represent the square root of the sum of the squared coefficient amplitudes of the harmonics of different orders. By comparing continuous and dashed lines, one can verify that high degree harmonics are replaced by low degree harmonics as ka increases if the phase of the target pattern is ignored.

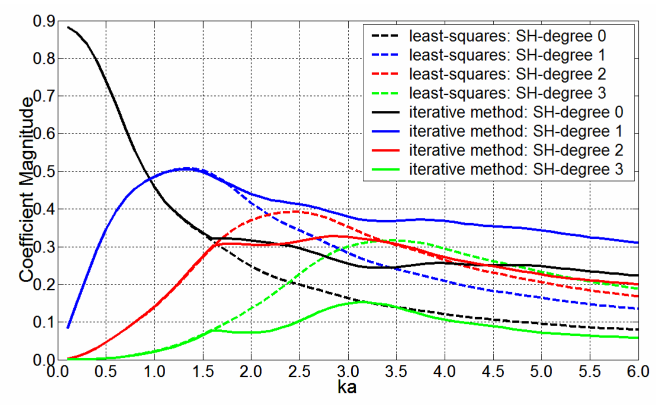

Fig.3 Spherical harmonic coefficients for N = 3.

The final magnitude errors - calculated by Eq.(7) – due to SHD truncation and to spherical array synthesis are shown in Fig.4 for different N. If phase of target pattern is considered, comparison of Figs.2 and 4 shows that SHD error is more important than array synthesis error. If phase of target pattern is ignored, SHD error prevails at low ka values and array synthesis error prevails at high ka values. This occurs since high degree harmonics become more important as ka increases, and they are difficult to be synthesized by a spherical array as shown in Fig.1. For ka > 3 approximately, it is better to use N = 1 than N = 3.

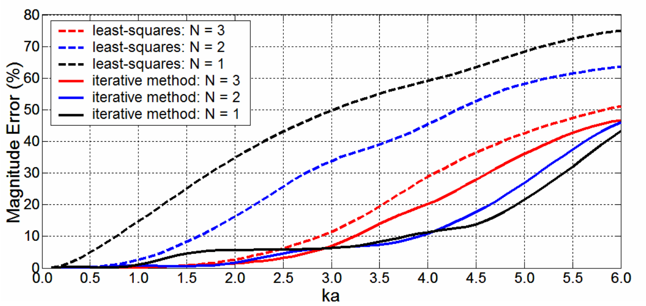

Fig.4 Final normalized magnitude errors – SHD with different truncation numbers and optimization procedures.

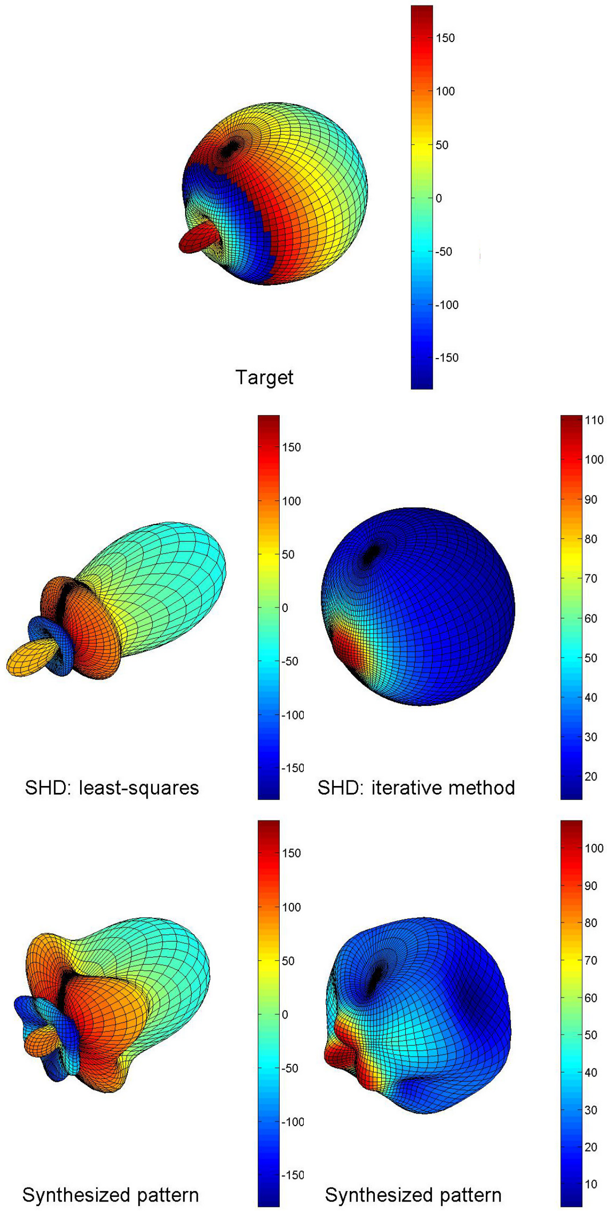

Fig.5 Magnitude and phase of the target pattern, its SHD (problem (IV) with N = 3 and problem (V) with N = 1) and the synthesized directivities at ka = 4.

Figure 5 shows the following directivity patterns evaluated for ka = 4: target pattern, patterns obtained after SHD by solving problem (IV) with N = 3 and problem (V) with N = 1, and decomposed patterns synthesized by the spherical loudspeaker array. Figure shapes indicate the function magnitudes and figure colors indicate function phase in degrees.

By representing a sound field with low degree harmonics, it is possible to reduce the power of the signals that must feed the drivers. Figure 6 shows the velocity magnitude that must be transmitted to one driver of the icosahedral array in order to synthesize its own directivity; the velocities of the other drivers are not presented, but their magnitudes are smaller. Theses values lead to a sound pressure level of 80dB in the main radiation direction for all ka and at a distance r = 20a from the center of the spherical array.

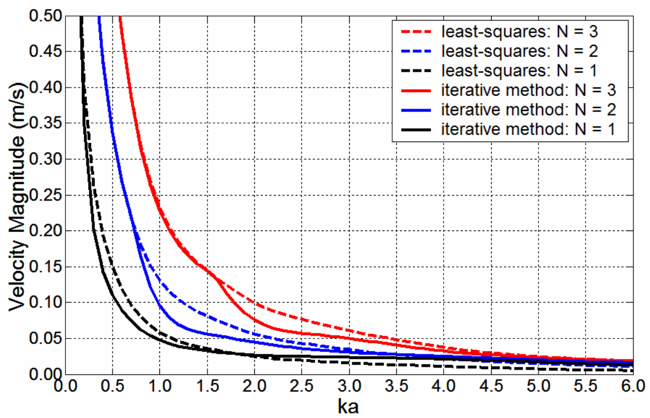

Fig.6 Velocity magnitude of the most solicited driver for different approaches.

5 Conclusion

If only magnitude response is required, the task of finding the best approximation of a complex function in a finite dimension subspace can be carried out by a simple iterative optimization method based on the least-squares. Although this method does not ensure the optimality of solutions, it performs better than the least-squares method if the algorithm is properly initialized. In this work, the method was illustrated on directivity synthesis by spherical loudspeaker arrays. The improvements in the directivity synthesis that can be achieved by ignoring the phase of the target pattern were discussed.

It was shown that a target pattern can be described by spherical harmonic functions of lower degrees whether phase error is ignored, so that the signal powers of the drivers and the synthesis error are greatly reduced, providing means to improve directivity synthesis for applications requiring only magnitude response.

Acknowledgments

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