Viscous and thermal effects in acoustic radiation problems

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In this paper we present a comprehensive analytical model for the acoustic propagation in a thermo-viscous fluids. Developement is done in two steps. First a complete model for air is realized and then a generalized model is shown, applicable to a any fluid. A new set of differential operators is presented in the course of this model. The analytical solution is derived from matrices of the system giving rise to the eigenwaves. For simple radiation problems one must apply the sommerfeld criteria to discard those eigenwaves which render the solution infinite. Classical quantities like pressure, temperature and the particle velocities are obtained afterwards.

1 Introduction

The conventional acoustic radiation is presented with the pressure variation in terms of the normal particle velocity, the Euler's equation. Temperture changes and the shearing action of the fluid is neglected owing to its influence just within the boundary layer. Such assumption is valid in open spaces, but need be reconsidered in closed spaces where the gaps might be of the order of the boundary layers. A very initial work was introduced by Gustav Kirchhoff in 1868 [1], where he made use of the Navier-Stokes equation and Fourier Law of heat conduction to generate a model of propagation in thermo-viscous fluids. Around this time, other domains such as thermo-acoustic refrigration and engines produced researches like [2] & [3]. In 50s and 60s some important studies were made on the damping of machine vibrations by thin films such as [4] & [5]. Work on miniaturized acoustic sensors also made use of such models as seen in [6] & [7]. Some relatively new researches from these fields combined include [8], [9] and [10]. Only very few of thses works could be termed comprehensive like [9] and [10] with respect to the four fundamental factors, namely, conductivity, viscosity, compressibility and inertia. Simplifying assumptions are there nonetheless, including zero pressure gradient perpendicular to the fluid film, normal particle velocity considered zero against the tangential velocity and the tangential differential opperator considered zero against the normal differential operator. In comparison our model does not take these assumptions and caters for all the fundamental factors. We present the single plate model here while the double plate transmission is reserved for the conference.

2 Linearized Fundamental Equations

In this section are presented the fundamental equations in linearized form. Therefore, ($'$) represents the first order variation while ($_o$) signifies the mean value. The equations detailed are conservation of continuity, momentum, energy and entropy respectively. It is important to note that Eq (4) & Eq (5) are mutually exclusive and a complete system may be determined with any one of them.

$$\frac{\partial \rho^{'}}{\partial t} + \rho_o \nabla . \vec{u^{'}} = 0 \tag{1}$$

$$\rho_o \frac{\partial u_\tau^{'}}{\partial t} = -\frac{\partial p^{'}}{\partial \tau} + (\mu_B + \frac{1}{3}\ \mu)\frac{\partial}{\partial \tau}\ (\nabla . \vec{u^{'}}) + \mu \Delta \vec{u^{'}} \tag{2}$$

$$\rho_o \frac{\partial u_z^{'}}{\partial t} = -\frac{\partial p^{'}}{\partial z} + (\mu_B + \frac{1}{3}\ \mu)\frac{\partial}{\partial z}\ (\nabla . \vec{u^{'}}) + \mu \Delta \vec{u^{'}} \tag{3}$$

$$\rho_o \frac{\partial e^{'}}{\partial t} = -p_o(\nabla . \vec{u^{'}}) + \lambda \Delta T^{'} \tag{4}$$

$$\rho_o T_o \frac{\partial s^{'}}{\partial t} = \lambda \Delta T^{'} \tag{5}$$

Here ρ is density, $\vec{u}$ the particle velocity vector, $u_\tau^{'}$ the amplitude of the tangential velocity given as $\sqrt{u_x^{'2} + u_y^{'2}}$, u_z normal velocity, p pressure, μ_B bulk viscosity, μ coefficient dynamic viscosity, e energy, T temperature, λ thermal conductivity and s entropy. The operators used in Eq (1) to Eq (5) are defined by Eq (6).

$$\begin{cases} \frac{\partial}{\partial \tau} & = \frac{1}{k_f}\left(k_x \frac{\partial}{\partial x} + k_y \frac{\partial}{\partial y}\right) \\ \nabla & = \frac{1}{k_f}\left(k_x \frac{\partial}{\partial x} + k_y \frac{\partial}{\partial y}\right)\hat{\tau} + \frac{\partial}{\partial z}\ \hat{z} \\ \Delta & = \frac{1}{k_f^2}\left(k_x^2 \frac{\partial^2}{\partial x^2} + 2k_x k_y \frac{\partial^2}{\partial x \partial y} + k_y^2 \frac{\partial^2}{\partial y^2}\right) + \frac{\partial^2}{\partial z^2} \end{cases} \tag{6}$$

where $k_f = \sqrt{k_x^2 + k_y^2}$ and k_x and k_y represent the plate wavenumber in the x and y direction respectively and therefore k_f is the flexural wavenumber of the plate in vacuum.

3 Thermodynamic Relationships for Air

Using the equation of state, $p = \rho R T$ and the two property rule, following relationships for air could be obtained.

$$\rho^{'} = \frac{\gamma}{c_o^2}\left(p^{'} - \hat{\beta} T^{'}\right) \tag{7}$$

where γ is the ratio of the specific heats at constant pressure and constant volume respectively, defined as C_p/C_v, c_o is the adiabatic speed of sound in air and $\hat{\beta}$ is ratio of the equilibrium pressure and the equlibrium temperature at constant density defined as p_o/T_o. The specific internal energy is given as,

$$e^{'} = C_v T^{'} \tag{8}$$

Using the definition of the specific heat at constant pressure and the factors which have already been defined above, the entropy of the system is given as,

$$s^{'} = \frac{C_p}{T_o}\left(T^{'} - \frac{\gamma - 1}{\gamma\hat{\beta}}\, p^{'}\right) \tag{9}$$

4 General Thermodynamic Relationships for Fluids

In this section we will develope the general thermodynamic relationships for any fluid. In order to realize this we will make use of the two property rule again Using the definition of isothermal compressibility, isobaric expansivity and the Maxwell's relations, we may write for mass density of the system,

$$\rho^{'} = \kappa\rho_o\left(p^{'} - \frac{\alpha}{\kappa}\, T^{'}\right) \tag{10}$$

where κ is the coefficient of isothermal compressibility and α is the coefficient of isobaric expansivity. Using the first Tds relationship & the definition of specific heat at constant voulume, we obtain the relationship for specefic internal energy of the system,

$$e^{'} = \mathcal{L}_1 p^{'} + \mathcal{L}_2 T^{'} \tag{11}$$

where $\mathcal{L}_1$ & $\mathcal{L}_2$ are defined as,

$$\mathcal{L}_1 = \frac{\kappa p_o - \alpha T_o}{\rho_o}$$
$$\mathcal{L}_2 = \frac{\rho_o\kappa C_v + (\alpha T_o - \kappa p_o)\alpha}{\rho_o\kappa}$$

Finally using the definition of α, the isobaric expansivity, we obtain the entropy of the system,

$$s^{'} = \frac{C_p}{T_o}\, T^{'} - \frac{\alpha}{\rho_o}\, p^{'} \tag{12}$$

5 Eigenwaves for Air

We search for a solution of the form as given by Eq (13),

$$\begin{cases} p^{'} & = \mathcal{A}e^{j\omega t}e^{jk_x x + jk_y y + jk_z z} \\ T^{'} & = \mathcal{B}e^{j\omega t}e^{jk_x x + jk_y y + jk_z z} \\ u^{'}_{\tau} & = \mathcal{C}e^{j\omega t}e^{jk_x x + jk_y y + jk_z z} \\ u^{'}_{z} & = \mathcal{D}e^{j\omega t}e^{jk_x x + jk_y y + jk_z z} \end{cases} \tag{13}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ are the arbitrary constants representing the unknowns of the system, ω is the angular frequency and k_z denotes the reduced wavenumber relating the acoustic wavenumber $k = \omega/c_o$ with the flexural wavenumber of the plate k_f. Using Eq (13) in the set Eq (1) to Eq (5) (excludig Eq (4)), Eq (7) & Eq (9), we obtain the homogeneous matrix for acoustic propagation in thermo-viscous air. Setting the determinant of this matrix equal to zero and solving for k_z returns six eigenwaves as shown in Fig. 1 where the real and imaginary parts of the eigenwaves are shown. We now invoke the Sommerfeld criteria on these eigenwaves to retain k_{z1}, k_{z3} & k_{z6} as the valid finite eigenwaves of acoustic propagation problem in an infinite thermo-viscous medium

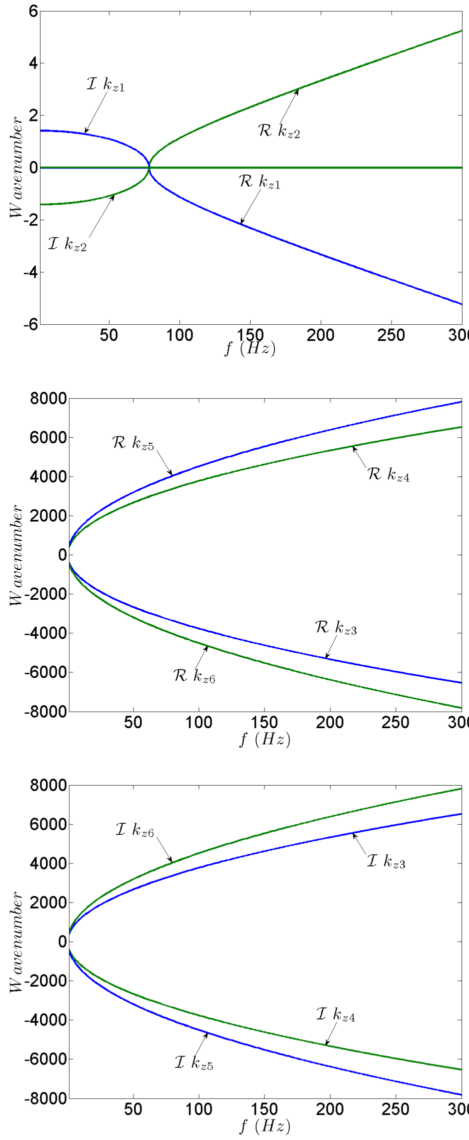

Figure 1: (a) Real & Imaginary Parts of the Eigenwaves k_{z1} & k_{z2}. (b) Real Parts of the Eigenwaves k_{z3}, k_{z4}, k_{z5} & k_{z6}. (c) Imaginary Parts of the Eigenwaves k_{z3}, k_{z4}, k_{z5} & k_{z6}.

6 Results for Air

The retained eigenwaves are resubstituted into the system matrix shown in Appendix A to render three matrices corresponding to the eigenwaves 1, 3 and 6 respectively. A homogeneous solution to these matrices may be obtained by normalizing any one of the arbitrary constants in §5 by setting it equal to 1. For three matrices, we will have three sets of solutions in terms of $\mathcal{A}$, $\mathcal{B}$ & $\mathcal{D}$. From these solutions we may write our final system shown by Eq (14).

$$\begin{bmatrix} \mathcal{B}_1 & \mathcal{B}_3 & \mathcal{B}_6 \\ 1 & 1 & 1 \\ \mathcal{D}_1 & \mathcal{D}_3 & \mathcal{D}_6 \end{bmatrix} \begin{bmatrix} \Gamma_1 \\ \Gamma_3 \\ \Gamma_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ U \end{bmatrix} \qquad (14)$$

Note that in the above equations $\mathcal{C}_1 = \mathcal{C}_3 = \mathcal{C}_6 = 1$. The unknowns of Eq (14) are the arbitrary constants Γ_1, Γ_3 & Γ_6 which depend on the boundary conditions at the air-plate interface and are determined by them. These are shown with the help of Eq (15).

$$\begin{cases} T'(z=0) & = 0 \\ u'_\tau(z=0) & = 0 \\ u'_z(z=0) & = Ue^{j(k_x x + k_y y)} \end{cases} \qquad (15)$$

The instantaneous temperature difference at the wall is zero and so is the amplitude of tangential velocity translating a no-slip condition at the wall. The normal particle velocity at the interface is equal to the normal plate velocity. The solution of the system permits us to determine the pressure, temperature, normal and tangential velocities and the radiation factor as demonstrated in Fig. 2

7 Results for Water

Acoustic radiation in water is modeled using the generalized relationships of §4 and in particular Eq (10) & Eq (12). The system matrix is shown in Appendix B. Following the steps as outlined in §5 & §6, we obtain Fig. 3 for water.

8 Discussion

The data used for air is; $T_o = 300\ K$, $p_o = 101325\ Pa$, $\rho_o = 1.1797600\ kg/m^3$, $c_o = 347.4\ m/s$, $\mu = 0.000018199\ kg/ms$, $\lambda = 0.026197599\ W/mK$, $C_p = 1005.458757\ J/kgK$, $\gamma = 1.4$, $\hat{\beta} = 337.5\ Pa/K$, $k_x = 1$, $k_y = 1$, $\mu_B = 1/3\mu$ & $U = 0.1\ m/s$. From Fig. 2 (a) & (b) we may deduce the thermal and viscous boundary layer respectively. Compared to the respective analytical form $\delta_{th} = \sqrt{2\lambda/\rho_o\omega C_p}$ and $\delta_{vs} = \sqrt{2\mu/\rho_o\omega}$ at the critical frequency ($f_c = c_o\sqrt{k_x^2 + k_y^2/2\pi}$), the results $\delta_{th} = 300\ \mu m$ and $\delta_{vs} = 250\ \mu m$ are in good agreement. In Fig. 2 (e) a small circle is shown around the critical frequency. In an ideal fluid, the radiation factor σ is strictly zero, but in a thermo-viscous fluid we find a finite but relatively smaller value around the circle shown. The data used for water is; $T_o = 300\ K$, $p_o = 101325$

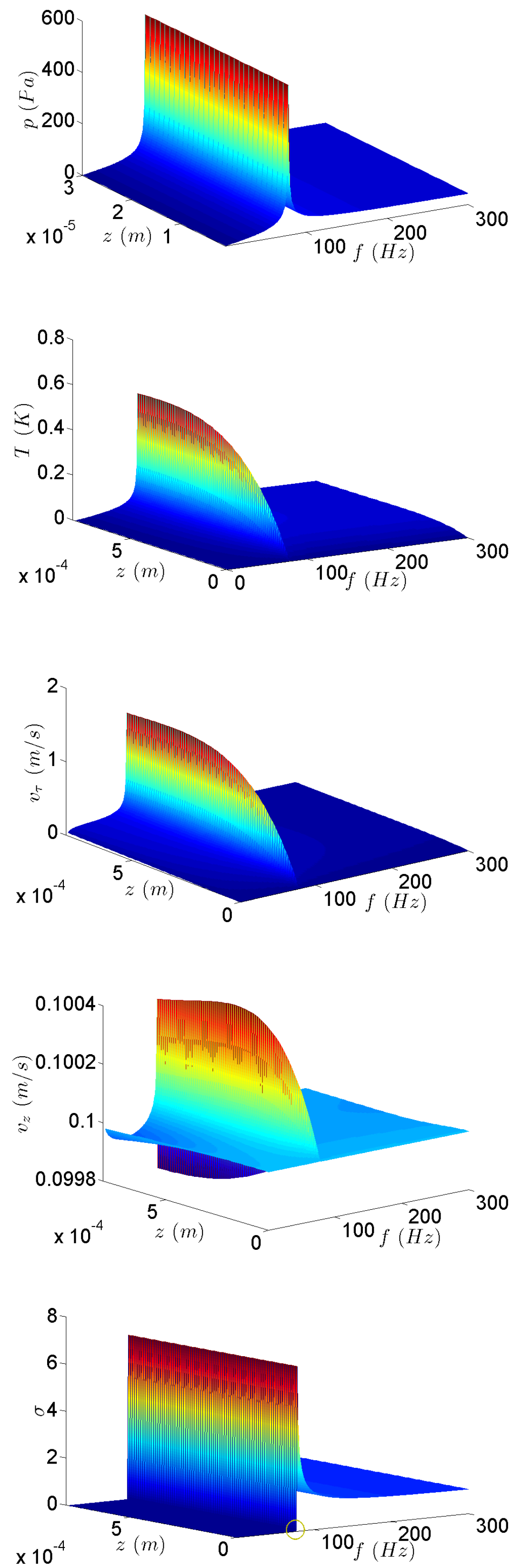

Figure 2: (a) Pressure. (b) Temperature. (c) Tangential Velocity. (d) Normal Velocity. (e) Radiation Factor.

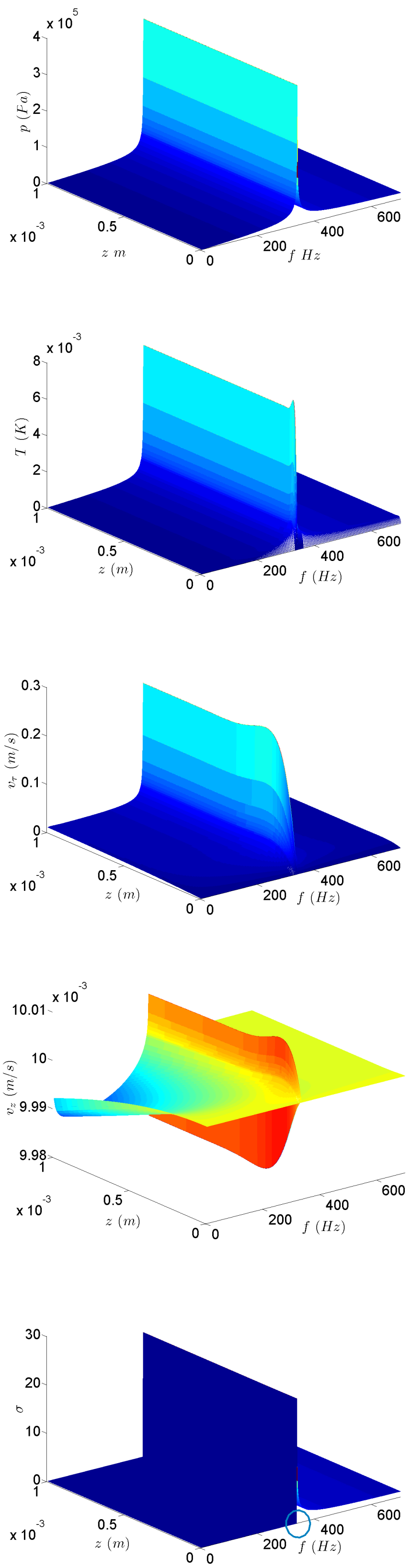

Figure 3: (a) Pressure. (b) Temperature. (c) Tangential Velocity. (d) Normal Velocity. (e) Radiation Factor.

Pa, $\rho_o = 996.55\ kg/m^3$, $c_o = 1501.9\ m/s$, $\mu = 8.5382\text{e-}3\ kg/ms$, $\lambda = 0.61032\ W/mK$, $C_p = 4.1806\text{e}3\ J/kgK$, $\kappa = 0.45106\text{e-}9$, $k_x = 1$, $k_y = 1$, $\mu_B = 3.4\mu$ & $U = 0.01\ m/s$. From Fig. 3 (a) & (b) we may deduce the thermal and viscous boundary layer respectively. It may be noted that the thermal boundary layer is much smller than the viscous boundary layer. In Fig. 3 (e) a small circle is shown around the critical frequency to mark some radiation in the usually non radiating zone which is very small but exists nonetheless. The presence of thermo-viscous effects cause a slight radiation tendency in the non-radiating zone which is marked by region below the critical frequency. This fact could be confirmed with Fig. 2 (d) & Fig. 3 (d) where the normal velocity is seen to decrease below the critical frequency which means an increase in the radiation. It could be demonstrated that using the same model on the double wall acoustic transmission we may obtain interseting gains in transmission loss as much as 5 dB. This demonstration is kept for the conference.

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Appendix A

The homogeneous system matrix for air is obtained by substituting the thermodynamic relationships for air in the linearized fundamental equations. Here $\mathcal{MU}$ denotes the quantity $\mu_B + \mu/3$. Once again we note that Eq (4) (energy) & Eq (5) (entropy) are mutually exclusive and so any one of them could be used. We employ Eq (5) as it generates a simple matrix. The numerical results would be exactly the same if had chosen Eq (4).

$$\mathcal{P} = \begin{bmatrix} j\frac{\gamma\omega}{c_o^2} & j\frac{\gamma\hat{\beta}\omega}{c_o^2} & jk_f\rho_o & jk_z\rho_o \\ jk_f & 0 & j\omega\rho_o + \mathcal{MU}(k_f^2) + \mu(k_f^2+k_z^2) & \mathcal{MU}(k_f k_z) \\ jk_z & 0 & \mathcal{MU}(k_f k_z) & j\omega\rho_o + \mathcal{MU}(k_z^2) + \mu(k_f^2+k_z^2) \\ j\frac{\omega\rho_o C_p(\gamma-1)}{\gamma\hat{\beta}} & -\{j\omega\rho_o C_p + \lambda(k_f^2+k_z^2)\} & 0 & 0 \end{bmatrix} \tag{A-1}$$

Appendix B

The homogeneous system matrix for water is obtained by substituting the general thermodynamic relations for fluids into the linearized fundamental equations.

$$\mathcal{P} = \begin{bmatrix} j\omega\kappa & -j\omega\alpha & jk_f & jk_z \\ jk_f & 0 & j\omega\rho_o + \mathcal{MU}(k_f^2) + \mu(k_f^2+k_z^2) & \mathcal{MU}(k_f k_z) \\ jk_z & 0 & \mathcal{MU}(k_f k_z) & j\omega\rho_o + \mathcal{MU}(k_z^2) + \mu(k_f^2+k_z^2) \\ j\omega\alpha T_o & -\{j\omega\rho_o C_p + \lambda(k_f^2+k_z^2)\} & 0 & 0 \end{bmatrix} \tag{B-1}$$

The unknowns of both these systems are the arbitrary constants $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$. These systems are used to find the eigenwaves as well as the arbitrary constants. The arbitrary constants are found by normalizing any one of them, i.e., setting it equal to one. For numerical stability reasons, this is chosen as $\mathcal{C}$.