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Practical modeling of acoustic losses in air due to heat conduction and viscosity

René Christensen[a], Peter Juhl[b] and Vicente Cutanda Henriquez[b]

[a]Oticon A/S, Kongebakken 9, 2765 Smoerum, Denmark
[b]Institute of Sensors, Signals and Electrotechnics, University of Southern Denmark, Niels Bohrs Allé 1, 5230 Odense S, Denmark
rch@oticon.dk

Accurate acoustic models of small devices with cavities and narrow slits and ducts should include the so–called boundary layer attenuation caused by thermal conduction and viscosity. The purpose of this paper is to present and compare different methods for including these loss mechanisms in analytical and numerical models. Two test cases with circular geometry have been used as references and are investigated both through measurements and the different models.
Four simulation methods are compared. The transmission line model is an analytical model which can be modified to include loss. Additionally, three numerical models have been tested. Two different implementations of the so-called Full Navier–Stokes model, one using the commercial package COMSOL Multiphysics, the other using Boundary Element code specifically aimed at the test case, are considered. The third numerical method, the so–called Low reduced frequency model, is evaluated using the commercial package ACTRAN.

1 Introduction

Sound propagation in fluids is usually described adequately by a lossless wave equation. However, when the fluid is confined in small slits, narrow tubes or similar situations, loss must be included for accurate modeling. The two main loss mechanisms are (i) thermal conduction loss caused mainly by the diffusion of heat from the sound field into the boundaries and (ii) viscosity loss as the fluid experiences friction at the boundaries [1, 2, 3, 4].

The loss mechanisms are predominantly important in a boundary layer with a frequency–dependent thickness. The relationships between acoustic wavelength, the boundary layer thickness and a characteristic length in the domain, for example the radius of a tube, must be considered when choosing calculation methods.

Two test cases have been considered. One is a closed cylindrical tube, the other consists of two circular cylindrical volumes connected by a circular cylindrical tube. The sound pressure level has been calculated at one end with a velocity boundary condition applied at the other end. The visco–thermal loss modifies the response around a characteristic resonance compared to a lossless case and thus the effect of including loss in the different models can be compared.

One analytical method and three numerical methods have been compared. The transmission line model is a proven analytical model which can include loss. Two of the numerical methods are based on the full set of governing equations for sound propagation. The full Navier–Stokes model is implemented in COMSOL Multiphysics[1] which is a Finite Element Method (FEM) simulation tool capable of working with multiple (and coupled) physical models. An alternative formulation of the full model is given by Bruneau *et al.* [5]. Here the governing equations are rewritten into a set of equations suited for implementation in e.g. a Boundary Element Method (BEM) environment. Such a formulation was developed by Karra [6], but terms related to the viscous loss were not included. Cutanda [7] later developed a formulation which takes all terms into account for a specific geometry. The implementation was done in a direct collocation BEM software package [8]. In the present work a similar model tailor suited for a particular test case was developed. The last model considered is the so–called Low reduced frequency model proposed by Beltman [9]; a somewhat simplified model compared to the full ones.

[1]Web page: http://www.comsol.com

Measurements have been carried out for one of the test cases using a microphone and a loudspeaker designed for hearing aids. The parameters of these are known and have been included in one of the models for a direct comparison to the measurements.

2 Methods

One analytical and three numerical methods for including loss in acoustic calculations have been considered. Some general assumptions are made for all of the models:

- The medium is homogeneous and in equilibrium.
- There is no mean flow.
- The particle displacement is low enough to ensure that nonlinear terms can be omitted.
- The dimensions and the wavelength are large compared to the molecular mean free path.

2.1 Transmission line model

The test cases with their cylindrical geometry can be modeled as transmission lines including both viscosity and thermal effects. The theory for the propagation of sound in a rigid cylindrical tube with visco–thermal loss was described by Kirchhoff [10]. The results and approximations have been discussed in various papers [11, 12, 13] and also the transmission line parameters describing the sound propagation can be found here.

The series impedance Z and the shunt admittance Y are expressed as

$$Z = j\left(\frac{\omega\rho_0}{\pi a^2}\right)(1-F_v)^{-1} \tag{1}$$

and

$$Y = j\left(\frac{\omega\pi a^2}{\rho_0 c^2}\right)(1+(\gamma-1)F_t), \tag{2}$$

with the radius a, the angular frequency ω, the static density ρ_0, the sound speed c and

$$F_v = \frac{2}{r_v\sqrt{-j}}\frac{J_1\left(r_v\sqrt{-j}\right)}{J_0\left(r_v\sqrt{-j}\right)}, \tag{3}$$

$$F_t = \frac{2}{r_t\sqrt{-j}}\frac{J_1\left(r_t\sqrt{-j}\right)}{J_0\left(r_t\sqrt{-j}\right)}, \tag{4}$$

$$r_v = a\sqrt{\frac{\rho\omega}{\mu}} \tag{5}$$

and

$$r_t = \sigma r_v, \tag{6}$$

with the square root of the Prandtl number

$$\sigma = \sqrt{\frac{\mu C_P}{\lambda}}, \tag{7}$$

with the thermal conductivity λ, the shear viscosity μ and J_m the Bessel function of the first kind, order m.

Once the series impedance Z and the shunt admittance Y have been determined the characteristic impedance Z_0 and the propagation wavenumber Γ of the transmission line are found as

$$Z_0 = \sqrt{\frac{Z}{Y}} \tag{8}$$

and

$$\Gamma = \sqrt{ZY}. \tag{9}$$

From these the transmission matrix linking the input and output variables, i.e. the pressure p and the particle velocity $\mathbf{v}$, can be established. The general equation for a transmission line with a length L is

$$\begin{pmatrix} p_i \\ v_i \end{pmatrix} = \begin{pmatrix} \cosh(\Gamma L) & Z_0 \sinh(\Gamma L) \\ Z_0^{-1} \sinh(\Gamma L) & \cosh(\Gamma L) \end{pmatrix} \begin{pmatrix} p_o \\ v_o \end{pmatrix}, \tag{10}$$

where subscripts i and o denote input and output, respectively.

Programs were made in MATLAB[2] which could be used for calculating the response for the transmission line models.

2.2 Full Navier–Stokes model

The governing equations for sound propagation in a viscous medium are the linearized Navier–Stokes equation

$$\rho_0 \frac{\partial \mathbf{v}}{\partial t} = -\nabla p + \left(\frac{4}{3}\mu + \eta\right) \nabla(\nabla \cdot \mathbf{v}) - \mu \nabla \times (\nabla \times \mathbf{v}), \tag{11}$$

the equation of continuity

$$\rho_0 \nabla \cdot \mathbf{v} + \frac{\partial \rho}{\partial t} = 0, \tag{12}$$

the energy equation

$$\rho_0 C_P \frac{\partial T}{\partial t} = \lambda \Delta + \frac{\partial p}{\partial t} \tag{13}$$

and the equation of state for an ideal gas

$$p = \rho R_0 T, \tag{14}$$

with the bulk viscosity η, the density variation ρ, the temperature variation T and R_0 equal to $C_P - C_V$ where C_P and C_V are the heat capacities per unit mass at constant pressure and constant volume, respectively.

This set of equations have been implemented in the commercial package COMSOL Multiphysics.

[2]Web page: http://www.mathworks.com

2.3 Boundary element model

An alternative full model is presented by Bruneau *et al.* [5] where the particle velocity is split into two; a rotational (or viscous) velocity $\mathbf{v}_v$ and a solenoidal (or laminar) velocity $\mathbf{v}_\ell$:

$$\mathbf{v} = \mathbf{v}_v + \mathbf{v}_\ell \tag{15}$$

The divergence of the viscous velocity is zero

$$\nabla \cdot \mathbf{v}_v = 0 \tag{16}$$

and the rotation of the laminar velocity is zero

$$\nabla \times \mathbf{v}_\ell = 0 \tag{17}$$

The sound pressure is also split:

$$p = p_a + p_h \tag{18}$$

where p_a is the acoustic pressure and p_h is the thermal pressure. Splitting the acoustic variables facilitates rewriting the governing equations into scalar wave equations for the acoustic and thermal pressures and a vector wave equation for the viscous velocity:

$$(\Delta + k_a^2) p_a = 0 \tag{19}$$

$$(\Delta + k_h^2) p_h = 0 \tag{20}$$

$$(\Delta + k_v^2) \mathbf{v}_v = \mathbf{0}. \tag{21}$$

The exact expressions for the acoustic wavenumbers k_a, the thermal wavenumber k_h and the viscous wavenumber k_v can be found in the literature [1, 5, 9].

For a general geometry in a cartesian coordinate system five similar wave equations must be solved for the two pressures and the velocity in the three directions. Additionally, the zero divergence of the viscous velocity must be included in the calculations. Each wave equation can in principle be solved using a boundary element formulation as described by Cutanda [7]. A matrix system can be build for e.g. the acoustic pressure:

$$\mathbf{A}_a \mathbf{p}_a - \mathbf{B}_a \frac{\partial \mathbf{p}_a}{\partial n} = 0, \tag{22}$$

where $\mathbf{A}_a$ and $\mathbf{B}_a$ are coefficient matrices, n is a normal vector and $\mathbf{p}_a$ is a vector with the acoustic pressure at nodes in a calculation mesh. Additionally, a set of boundary conditions must be ensured:

$$\overline{\mathbf{V}} = \phi_a \nabla p_a + \phi_h \nabla p_h + \mathbf{v}_v \tag{23}$$

and

$$T = \tau_a p_a + \tau_h p_h \tag{24}$$

where $\overline{\mathbf{V}}$ is a prescribed boundary velocity and with ϕ_a, ϕ_h, τ_a and τ_h being coefficients depending on both the medium characteristics and the frequency. One challenge with this procedure is to express the global boundary conditions in a way that fits the boundary element formulation. Two approaches are currently being considered; an axisymmetrical BEM code and a 3D BEM code. The axisymmetrical formulation is attractive in that the calculation mesh consists of line elements and therefore facilitates faster calculations. But in the derivation of the boundary element formulation one encounters the problem of the vector wave equation being split into three equations for the ρ, θ and z-direction with only one being a general scalar wave equation, namely for the z-direction. For the other two directions additional terms appear and these equations are not readily handled with the existing code. Instead, a 3D formulation was made where it was assumed that the variation in the θ-direction is zero and that the boundaries not having a velocity condition were rigid and isothermal. The walls with a velocity boundary condition have motion only in the normal direction, not in the tangential direction. With these assumptions an equation system, linking the acoustic pressure and the normal velocity boundary condition, can be assembled as:

$$\overline{\mathbf{V}}_n = \left(\phi_a \mathbf{B}_a^{-1}\mathbf{A}_a - \phi_h \frac{\tau_a}{\tau_h}\mathbf{B}_h^{-1}\mathbf{A}_h + \mathbf{A}_v^{-1}\mathbf{B}_v\mathbf{D}_v\right)\mathbf{p}_a. \tag{25}$$

The matrices with the subscripts a, h and v are related to the acoustic pressure, the thermal pressure and the viscous velocity in z-direction, respectively. $\mathbf{D}_v$ is a matrix which is effectively a constant times a second order derivative in the axial direction z.

2.4 Low reduced frequency model

The so–called low reduced frequency model has its starting point in the full Navier–Stokes model, but different scalings and assumptions reduce the complexity. The method has been described in detail by Beltman [9, 14]. Dimensionless small harmonic perturbations are introduced:

$$\overline{p} = p_0\left(1 + \hat{p}e^{j\omega t}\right) \tag{26}$$

$$\mathbf{v} = c\hat{\mathbf{v}}e^{j\omega t} \tag{27}$$

$$\overline{T} = T_0\left(1 + \hat{T}e^{j\omega t}\right) \tag{28}$$

$$\overline{\rho} = \rho_0\left(1 + \hat{\rho}e^{j\omega t}\right) \tag{29}$$

In addition a dimensionless gradient is introduced

$$\hat{\nabla} = \ell\nabla \tag{30}$$

as well as a dimensionless Laplace operator

$$\hat{\Delta} = \ell^2\Delta \tag{31}$$

where ℓ is a characteristic length scale; e.g. the radius of a tube.

Dimensionless parameters are introduced, such as the shear wavenumber

$$\hat{s} = \ell\sqrt{\frac{\rho_0\omega}{\mu}}, \tag{32}$$

the reduced frequency

$$\hat{k} = \frac{\omega\ell}{c} \tag{33}$$

and the viscosity ratio

$$\hat{\xi} = \frac{\eta}{\mu} \tag{34}$$

The low reduced frequency model assumes that the acoustic wavelength is large compared to the the length scale ℓ, so that $\hat{k} \ll 1$, and that the acoustic wavelength is large compared to the boundary layer thickness, i.e. $\hat{k}/\hat{s} \ll 1$. Under these assumptions the governing equations for harmonic perturbations are reduced to

$$j\hat{\mathbf{v}}^{pd} = -\frac{1}{\hat{k}\gamma}\hat{\nabla}^{pd}\hat{p} + +\frac{1}{\hat{s}^2}\hat{\Delta}^{cd}\hat{\mathbf{v}}^{pd} \tag{35}$$

where γ is the ratio of specific heats,

$$0 = -\frac{1}{\hat{k}\gamma}\hat{\nabla}^{cd}\hat{p}, \tag{36}$$

$$\hat{\nabla}\cdot\hat{\mathbf{v}} + j\hat{k}\hat{\rho} = 0, \tag{37}$$

$$\hat{p} = \hat{\rho} + \hat{T} \tag{38}$$

and

$$j\hat{T} = \frac{1}{\hat{s}^2\sigma^2}\hat{\Delta}^{cd}\hat{T} + j\left(\frac{\gamma-1}{\gamma}\right)\hat{p}, \tag{39}$$

where the propagational direction (pd) and cross–sectional directions (cd) have been separated.

The low reduced frequency model is implemented in the software package ACTRAN[3].

3 Test cases

The boundary element code was tested against a transmission line model for the simple case of a circular cylindrical and closed tube. The length of the tube is 0.17 m and two radii, 0.01 m and 0.001 m, were used. A normal velocity of 1 m/s was applied at one end of the tube and the sound pressure level at the other end was calculated. The frequency range was chosen so that the first axial resonance was apparent. The results are shown in figures 1 and 2, respectively.

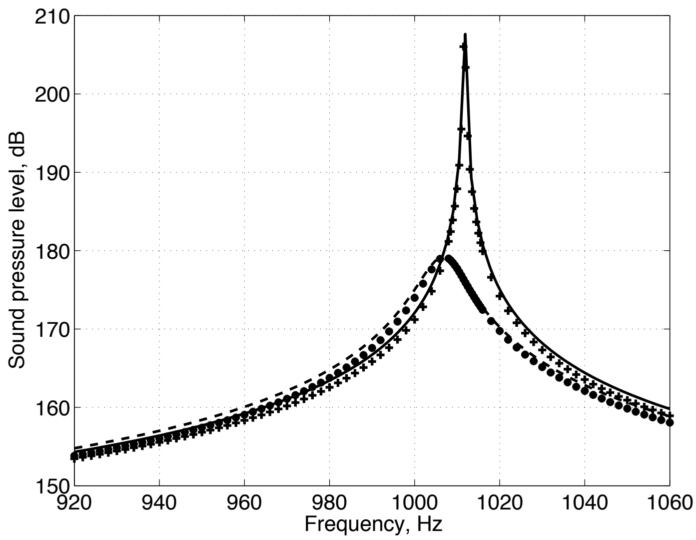

Figure 1: The sound pressure level for the tube with a radius of 1 cm. Transmission line model without loss (—) and with loss (- - -), and BEM model without loss (+) and with loss (•).

[3]Web page: http://www.fft.be

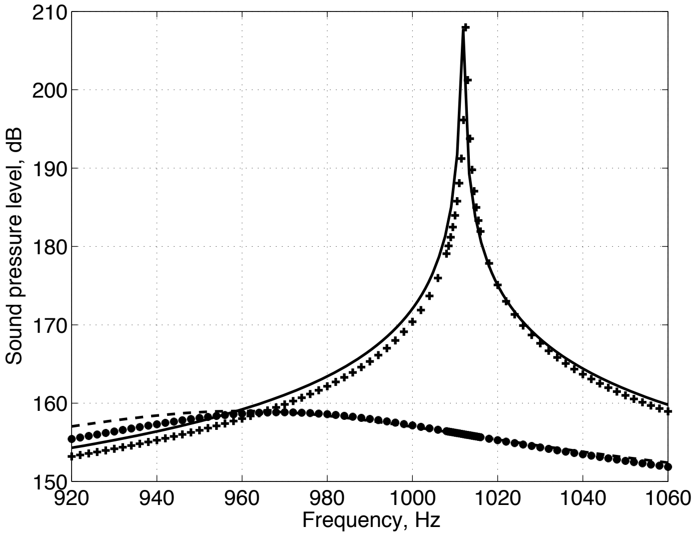

Figure 2: The sound pressure level for the tube with a radius of 1 mm. Transmission line model without loss (—) and with loss (- - -), and BEM model without loss (+) and with loss (•).

Another test case with circular, cylindrical geometry was used in a comparison between the transmission line model, a COMSOL Multiphysics calculation and an ACTRAN . Two volumes are connected by a tube. The loss in the tube is considered the primary contribution to the overall loss in the system, and therefore only the tube is lossy in the transmission line model and the ACTRAN model. The test case is shown in Fig. 3 and the dimensions are found in Table 1. The velocity at one end is

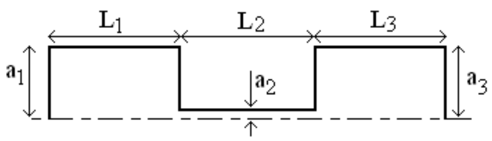

Figure 3: The geometry of the test case with the lengths L_1, L_2 and L_3, and the radii a_1, a_2 and a_3. There is rotational symmetry around the axis.

Symbol	Value	SI Units
$L_1 = L_3$	9.4×10^{-3}	m
L_2	1.0×10^{-2}	m
$a_1 = a_3$	5.0×10^{-3}	m
a_2	5.0×10^{-4}	m

Table 1: Test case dimension.

1 m/s and again the sound pressure level at the other end is calculated.

4 Measurements

As an addition to the simulations measurements were carried out. The second test case was constructed in aluminium and fitted with a loudspeaker at one end and a microphone at the other end. The measurement results have been compared to a modified transmission

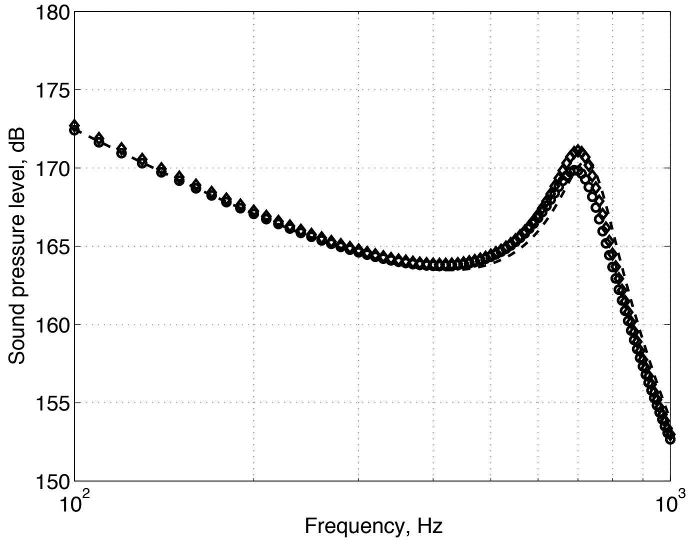

Figure 4: The sound pressure level for the test case in figure 3 with loss included using the transmission line model (- - -), ACTRAN (⋄) and COMSOL (◦).

line model where the characteristics of the transducers are included. The transmission line was chosen as a representative model whose results are close to those of the other models. A comparison of the results (Fig. 5) where the levels have been offset to get the most overlap of the curves shows that there is a good agreement between simulation and measurement. Discrepancies at low frequencies are due to limitations in the transducer model, and at high frequencies the dynamic range of the measurement setup limits the accuracy.

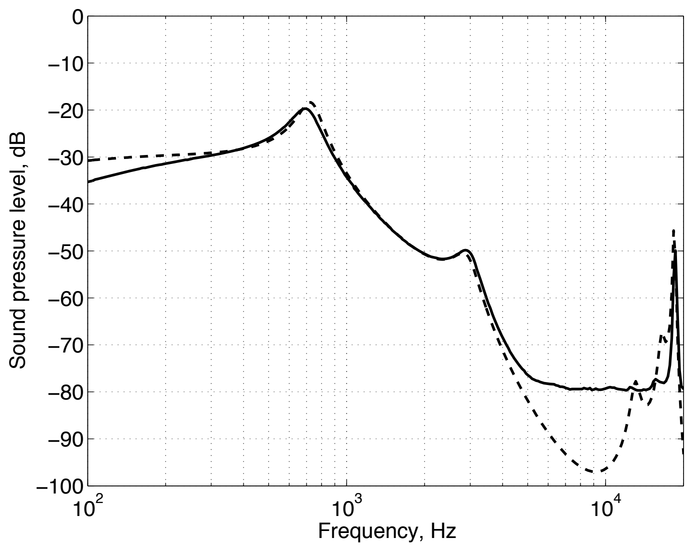

Figure 5: The sound pressure level for the transmission line model with the characteristics of the loudspeaker and the microphone included (- - -), and the measurement result (—) for the test case in figure 3.

5 Conclusion

The losses in acoustic due to heat conduction and viscosity can be incorporated in different models. A comparison between four different models, all including the visco–thermal losses, has been made.

The transmission line model gave results comparable to those of the numerical models. A transmission line model with transducer characteristics included was compared to measurements and the correspondence between the two was good. For rigid, cylindrical tubes with isothermal walls this model will give precise results and since the model is analytical, calculations can be carried out quickly.

The full Navier–Stokes model implemented in COMSOL Multiphysics gave results close to those of the transmission line model. Additionally, this full model is numerical and therefore calculations can be carried out for any geometry.

The alternative full Navier–Stokes model implemented in a Boundary Element environment lead to results very similar to those of the transmission line model. The code is aimed at a specific test case, but the model can be expanded to more general geometries.

The low reduced frequency model is a simplified version of the full model where it is assumed that the pressure in the cross-sectional direction is constant. The method is implemented in ACTRAN and calculations with the tube having visco–thermal loss have been carried out. The results are again very similar to the transmission line model and the full Navier–Stokes model.

Acknowledgments

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