Acoustics'08 Paris
June 29-July 4, 2008
www.acoustics08-paris.org

Robust array pattern synthesis with uncertain manifold vector

Shefeng Yan

Department of Electronics and Telecommunications, Norwegian University of Science and Technology, O.S. Bragstads plass 2A, 7491 Trondheim, Norway
sfyan@ieee.org

The knowledge of the array manifold vectors of an acoustic array can be imprecise, which is often the case in practice. This may cause undesirably high sidelobes for a nominal optimal beamformer where the array manifold vectors are assumed to be known exactly. Although the norm constraint on beamformer weights can be imposed to improve the robustness of the optimal beamformer, it is not clear how to choose the optimal constrained parameter based on the known level of uncertainty of the array manifold vectors. A pattern synthesis approach to arbitrary arrays with robustness against array manifold vectors errors is developed. Our technique optimizes the worst-case performance by minimizing the worst-case sidelobe level while maintaining a distortionless response to the worst-case signal steering vector. The parameters can be optimally chosen based on the uncertainty of the array manifold vector. The robust beamformer problem is shown to be convex, which can be efficiently solved using second-order cone programming. A simple lower bound on how much worse the robust optimal beamformer will be compared to the nominal one is also derived. Computer simulations show better performance of the proposed robust beamformer.

1 Introduction

Array signal processing has wide applications in sonar, radar, wireless communications, seismology, medical imaging, etc [1, 2]. Array pattern synthesis is one of the most important tasks. Beamformers can have unacceptably high sidelobes, which can lead to severe performance degradation in the case of unexpected or suddenly appearing interferers. The issue of synthesizing array patterns with low sidelobes has received much attention over the years.

The classical Dolph-Chebychev synthesis technique was given by Dolph [3], which results in a sidelobe level that is minimum possible for a given mainlobe width. However, this method can only be applied for uniform linear arrays with isotropic and equal element patterns. Many pattern synthesis approaches have been proposed for arrays of arbitrary geometry and with nonisotropic and unequal element patterns using adaptive array theory [4], recursive least squares algorithm [5], and so on. For both approaches in [4] and [5], an iteration process is required and the convergence of the iterations in general cannot be guaranteed. Array pattern synthesis approaches using convex optimization were presented in [6] and [7], where peak sidelobe level control designs were considered for deterministic and adaptive arrays, respectively. For both approaches, the convergence can be guaranteed. Recently, the author presented pattern synthesis methods for broadband arrays using convex optimization [8, 9], in which the broadband pattern sidelobes can be strictly controlled.

In all these approaches [3-9], the array manifold vectors are assumed to be known exactly. In practice, however, the knowledge of the array manifold vectors can be imprecise. This manifold vector mismatch can be caused by sensor sensitivity mismatch, channel gain and phase mismatch, element position perturbations, structural scattering, shadowing, mutual coupling between the sensors, and so on. This may cause undesirably high sidelobes for a nominal optimal beamformer.

The quadratically constrained beamformers (e.g., weight vector norm constrained beamforming method [10, 11], whose implementation is often based on the so-called diagonal loading of the covariance matrix [12]) are known to be able to improve the robustness of a beamformer. Although this constraint can be imposed to the aforementioned optimal array pattern synthesis approaches to improve their robustness, it is not clear how to choose the optimal constrained parameter based on the uncertainty of the array manifold vectors.

Recently, several robust approaches that make explicit use of an uncertainty set of the signal steering vector were proposed to adaptive beamforming [13-17]. These approaches also belong to the extended class of diagonal loading approaches, but the diagonal loading factor can be calculated based on the uncertainty set of the signal steering vector. Among them, the approach in [13] is based on the worst-case performance optimization. More recently, a worst-case robust beamforming approach with multiplicative uncertainty in the weights is proposed [18]. The obtained beamformer is quite robust to weight variation.

In this paper, a new powerful pattern synthesis approach to arbitrary arrays with low sidelobes and robustness against array manifold perturbations is developed. This approach is also based on the optimization of worst-case performance. It's shown that the parameters of this robust beamformer can be calculated based on the known level of uncertainty of the array manifold vectors. The robust array pattern synthesis problem is rewritten in an equivalent convex optimization form, which can be solved efficiently using a second-order cone programming (SOCP) solver such as SeDuMi [19].

2 Background

Consider an M-element array. Assuming that the arriving signal is a narrowband plane wave, the array manifold vector can be expressed as

$$\mathbf{v}(\theta) = [v_1(\theta),\ldots,v_m(\theta),\ldots,v_M(\theta)]^T$$
$$= [A_1(\theta)e^{j\phi_1(\theta)},\ldots,A_m(\theta)e^{j\phi_m(\theta)},\ldots,A_M(\theta)e^{j\phi_M(\theta)}]^T,$$
$$\theta \in \Theta, \qquad (1)$$

where $(\cdot)^T$ denotes the transpose, $v_m(\theta) = A_m(\theta)e^{j\phi_m(\theta)}$ is the element response of the mth sensor with $A_m(\theta)$ being the magnitude of the element pattern and $\phi_m(\theta)$ being the phase delay due to propagation. Θ is the set of all possible wave parameters. In the plane case, $\Theta = [0°, 360°]$ corresponds to the possible arrival angle of a plane wave.

The array pattern is a function of array's response to a unit input signal over angles of interest. It is given by

$$p(\theta) = \mathbf{w}^H \mathbf{v}(\theta), \ \theta \in \Theta, \qquad (2)$$

where $\mathbf{w} = [w_1, w_2, ..., w_M]^T$ is the complex weight vector of the array and $(\cdot)^H$ denotes the Hermitian transpose.

Let Θ_{SL} be the given sidelobe region and $\theta_1, ..., \theta_K \in \Theta_{SL}$ be the angular grids chosen that properly approximate the sidelobe region by a finite number of directions. The sidelobe level of the array pattern can be approximated fairly accurately by

$$G = \max_{k=1,...,K} |\mathbf{w}^H \mathbf{v}(\theta_k)|, \ \theta_k \in \Theta_{SL}, \tag{3}$$

The goal of a nominal optimal beamformer is to minimize the sidelobe level with the distortionless response constraint in the desired direction. The array pattern synthesis problem is formulated as follows:

$$\min_{\mathbf{w}} G, \text{ subject to } |\mathbf{w}^H \mathbf{v}(\theta_0)| = 1, \tag{4}$$

where θ_0 is the desired direction. The manifold vector corresponding to the desired signal, $\mathbf{v}(\theta_0)$, is referred to as the steering vector.

Eq.(4) can be transformed to an equivalent problem

$$\min_{\mathbf{w}} G, \text{ subject to } \mathbf{w}^H \mathbf{v}(\theta_0) = 1, \tag{5}$$

which is a convex problem [6]. The solution of Eq.(5) can be obtained using SOCP solver such as SeDuMi.

SOCP is a subclass of the well-structured convex programming problems where a linear function is minimized subject to a set of second-order cone constraints and possibly a set of linear equality constraints. The global optimal numerical solution of an SOCP problem is guaranteed if it exists. A review of the applications of SOCP was presented by Lobo et al. [20].

Let $\mathbf{w}_{nom}$ be the solution of optimization problem (5). Using Eq.(3), the resulting sidelobe level of this nominal beamformer is given by

$$G_{nom} = \max_{k=1,...,K} |\mathbf{w}_{nom}^H \mathbf{v}(\theta_k)|, \ \theta_k \in \Theta_{SL}. \tag{6}$$

3 Proposed method

In the practical case, only an imprecise knowledge of the actual manifold vectors is available. The actual manifold vector $\tilde{\mathbf{v}}(\theta_k) = [\tilde{v}_1(\theta_k), \tilde{v}_2(\theta_k), ..., \tilde{v}_M(\theta_k)]^T$ can therefore be expressed as

$$\tilde{\mathbf{v}}(\theta_k) = \mathbf{v}(\theta_k) + \mathbf{\Delta}(\theta_k), \ k = 0,1,...,K, \tag{7}$$

where $\mathbf{v}(\theta_k)$ is the ideal (presumed) array manifold vector, and $\mathbf{\Delta}(\theta_k)$ is an unknown complex vector that describes the array manifold vector distortions. The only knowledge we have about $\tilde{\mathbf{v}}(\theta_k)$ is that it belongs to an uncertainty set that we denote by $U(\theta_k)$ and will be defined later.

When there is uncertainty, the distortionless response constraint in Eq.(4) should be replaced with

$$|\mathbf{w}^H \mathbf{u}(\theta_0)| \geq 1, \text{ for all } \mathbf{u}(\theta_0) \in U(\theta_0). \tag{8}$$

The worst-case sidelobe level is

$$G_{wc} = \max_{\theta_k \in \Theta_{SL}} \max_{\mathbf{u}(\theta_k) \in U(\theta_k)} |\mathbf{w}^H \mathbf{u}(\theta_k)|. \tag{9}$$

Using the worst-case performance optimization, our robust formulation of optimal beamformer can be written as the following constrained minimization problem:

$$\min_{\mathbf{w}} G_{wc}, \text{ subject to } \min_{\mathbf{u}(\theta_0) \in U(\theta_0)} |\mathbf{w}^H \mathbf{u}(\theta_0)| \geq 1. \tag{10}$$

We assume that the array manifold vector distortion $\mathbf{\Delta}(\theta_k)$ is norm-bounded by some known constant $\varepsilon_k > 0$,

$$\| \mathbf{\Delta}(\theta_k) \| \leq \varepsilon_k. \tag{11}$$

Then, the actual array manifold vector $\tilde{\mathbf{v}}(\theta_k)$ belongs to the following uncertainty set:

$$U(\theta_k) \overset{\Delta}{=} \{\mathbf{u}(\theta_k) \quad \mathbf{u}(\theta_k) = \mathbf{v}(\theta_k) + \mathbf{e}(\theta_k), \| \mathbf{e}(\theta_k) \| \leq \varepsilon_k\},$$
$$k = 0,1,...,K. \tag{12}$$

Here, $U(\theta_k)$ is an ellipsoid that covers the possible range of values of $\tilde{\mathbf{v}}(\theta_k)$ due to imprecise knowledge of the array manifold $\mathbf{v}(\theta_k)$.

Observe that (see [13]), for any $\mathbf{u}(\theta_0) \in U(\theta_0)$, we have

$$\begin{aligned} |\mathbf{w}^H \mathbf{u}(\theta_0)| &= |\mathbf{w}^H \mathbf{v}(\theta_0) + \mathbf{w}^H \mathbf{e}(\theta_0)| \\ &\geq |\mathbf{w}^H \mathbf{v}(\theta_0)| - |\mathbf{w}^H \mathbf{e}(\theta_0)| \\ &\geq |\mathbf{w}^H \mathbf{v}(\theta_0)| - \varepsilon_0 \| \mathbf{w} \|. \end{aligned} \tag{13}$$

Note that we require that $|\mathbf{w}^H \mathbf{v}(\theta_0)| \geq \varepsilon_0 \| \mathbf{w} \|$. Moreover, the equality in Eq.(13) holds with the choice of

$$\mathbf{e}(\theta_0) = -\varepsilon_0 (\mathbf{w} / \| \mathbf{w} \|) e^{j \arg[\mathbf{w}^H \mathbf{v}(\theta_0)]}. \tag{14}$$

Therefore, we have

$$\min_{\mathbf{u}(\theta_0) \in U(\theta_0)} |\mathbf{w}^H \mathbf{u}(\theta_0)| = |\mathbf{w}^H \mathbf{v}(\theta_0)| - \varepsilon_0 \| \mathbf{w} \|. \tag{15}$$

Similarly, with $\mathbf{e}(\theta_k) = \varepsilon_k (\mathbf{w} / \| \mathbf{w} \|) e^{j \arg[\mathbf{w}^H \mathbf{v}(\theta_k)]}$, we have

$$\max_{\mathbf{u}(\theta_k) \in U(\theta_k)} |\mathbf{w}^H \mathbf{u}(\theta_k)| = |\mathbf{w}^H \mathbf{v}(\theta_k)| + \varepsilon_k \| \mathbf{w} \|. \tag{16}$$

Substituting Eq.(16) into Eq.(9) gives

$$G_{wc} = \max_{\theta_k \in \Theta_{SL}} [|\mathbf{w}^H \mathbf{v}(\theta_k)| + \varepsilon_k \| \mathbf{w} \|]. \tag{17}$$

Using Eq.(15) and Eq.(17) in Eq.(10) gives

$$\min_{\mathbf{w}} \max_{k=1,...,K} [|\mathbf{w}^H \mathbf{v}(\theta_k)| + \varepsilon_k \| \mathbf{w} \|],$$
$$\text{subject to } |\mathbf{w}^H \mathbf{v}(\theta_0)| - \varepsilon_0 \| \mathbf{w} \| \geq 1. \tag{18}$$

Using the fact that the cost function in Eq.(18) is unchanged when $\mathbf{w}$ undergoes an arbitrary phase rotation, it can be written as

$$\min_{\mathbf{w}} \max_{k=1,...,K} [|\mathbf{w}^H \mathbf{v}(\theta_k)| + \varepsilon_k \| \mathbf{w} \|],$$
$$\text{subject to } \mathbf{w}^H \mathbf{v}(\theta_0) \geq 1 + \varepsilon_0 \| \mathbf{w} \|. \tag{19}$$

This is also a convex problem and its solution, $\mathbf{w}_{rob}$, can be obtained using an SOCP solver such as SeDuMi.

We next derive a simple lower bound on how much worse the robust optimal beamformer will be compared to the nominal beamformer when $\varepsilon_k = \varepsilon$, $k = 0,1,...,K$.

Let G_{wco} be the optimal value of the robust beamforming problem (19), i.e.,

$$G_{wco} = \max_{k=1,\ldots,K} [|\mathbf{w}_{rob}^H \mathbf{v}(\theta_k)| + \varepsilon \|\mathbf{w}_{rob}\|]$$
$$= \max_{k=1,\ldots,K} |\mathbf{w}_{rob}^H \mathbf{v}(\theta_k)| + \varepsilon \|\mathbf{w}_{rob}\| . \quad (20)$$

Note that an equivalent problem to Eq.(15) is

$$\min_{\mathbf{w}} G \text{, subject to } \mathbf{w}^H \mathbf{v}(\theta_0) \geq 1, \quad (21)$$

This can be proved by contradiction as follows. If the minimum of the objective function in Eq.(21) is achieved when $p_0 \overset{\Delta}{=} \mathbf{w}^H \mathbf{v}(\theta_0) > 1$, replacing $\mathbf{w}$ with $\mathbf{w}/p_0$, we can decrease the objective function while the inequality constraint will be still satisfied. Therefore, optimization problem (21) is equivalent to Eq.(5).

Note that $\mathbf{w}_{rob}$ is a feasible point of the nominal problem (21), by the optimality definition, we have

$$\max_{k=1,\ldots,K} |\mathbf{w}_{rob}^H \mathbf{v}(\theta_k)| \geq \max_{k=1,\ldots,K} |\mathbf{w}_{nom}^H \mathbf{v}(\theta_k)|, \quad (22)$$

From the constraint in Eq.(19), we have

$$\varepsilon \|\mathbf{w}_{rob}\| \leq \mathbf{w}_{rob}^H \mathbf{v}(\theta_0) - 1 = \|\mathbf{w}_{rob}\| \cdot \|\mathbf{v}(\theta_0)\| - 1 \quad (23)$$

and hence, we get a lower bound on $\|\mathbf{w}_{rob}\|$:

$$\|\mathbf{w}_{rob}\| \geq 1/(\|\mathbf{v}(\theta_0)\| - \varepsilon). \quad (24)$$

Using Eq.(6), Eq.(20), Eq.(22) and Eq.(24) gives

$$G_{wco} - G_{nom} \geq \varepsilon / (\|\mathbf{v}(\theta_0)\| - \varepsilon). \quad (25)$$

For example, if $\varepsilon = 0.05\|\mathbf{v}(\theta_0)\|$, which corresponds to 5% uncertainty in the manifold vectors, then

$$G_{wco} - G_{nom} \geq 0.0526. \quad (26)$$

In particular, we cannot achieve a worst case sidelobe level smaller than $20\log_{10} 0.0526 = -25.6$ dB, regardless of the array geometry or the number of elements. However, the resulting sidelobe level, $\max_{\theta_k \in \Theta_{SL}} |\mathbf{w}^H \tilde{\mathbf{v}}(\theta_k)|$, can be smaller than that value. Interestingly, this l_2-regularization of robust pattern synthesis problem has the same lower bound on worst case sidelobe level as the l_1-regularization of beamforming problem [18] with no manifold vectors uncertainty and 5% weight uncertainty.

4 Numerical results

Consider a 24-element uniform circular array with radius $r = 0.96\lambda$, where λ is the wavelength. The location of the mth sensor is

$$(x_m, y_m) = (r\cos[\frac{2\pi}{24}(m-0.5)], r\sin[\frac{2\pi}{24}(m-0.5)]).$$

Assume that the mth element response is given by

$$v_m(\theta) = e^{j2\pi[x_m \cos(\theta) + y_m \cos(\theta)]/\lambda}. \quad (27)$$

Clearly, $\|\mathbf{v}(\theta)\| = \sqrt{M}$.

We provide numerical examples in this section to compare the performances of the delay-and-sum beamformer, the nominal optimal beamformer (5), and the robust optimal beamformer (19). We use $\varepsilon_k = 0.05\sqrt{M}$ in Eq.(19) and let $\theta_0 = 180°$ and $\Theta_{SL} = [0°,155°] \cup [205°,360°]$ which is sampled with $1°$.

Fig. 1 shows the ideal beam pattern obtained by the three beamformers when there is actually no array manifold error, i.e., $\tilde{\mathbf{v}}(\theta_k) = \mathbf{v}(\theta_k)$. It is seen that the obtained sidelobe level of delay-and-sum beamformer is about -7.9 dB, which can be prohibitively high in many applications. The nominal optimal beamformer provides excellent sidelobes when there are no array manifold errors. The resulting sidelobe level of the robust beamformers is about -24.3 dB, lower than delay-and-sum beamformer and higher than nominal optimal beamformer. By calculating, the worst case sidelobe level of the robust beamformer is $G_{wco} = -16.8$ dB.

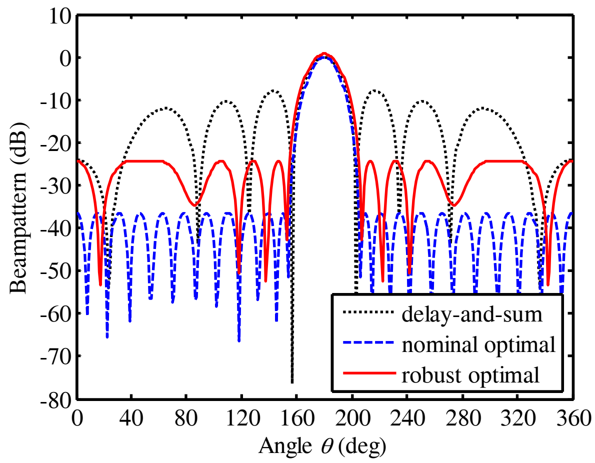

Fig. 1 Beampattern when there is no manifold error.

We next consider a scenario with the array manifold uncertainty. Assume that each element of the manifold vector for each direction is perturbed with a zero-mean circularly symmetric complex Gaussian random variable normalized so that $|\tilde{v}_m(\theta_k) - v_m(\theta_k)| = 0.05$. The perturbing Gaussian random variables are independent of each other. We use 100 Monte Carlo simulations to compare the statistical performances of the resulting sidelobe level of these three beamformers, as shown in Fig. 2.

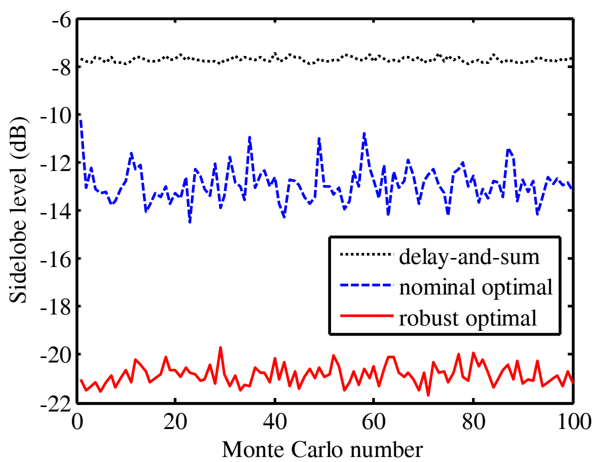

Fig. 2 Sidelobe level when $|\tilde{v}_m(\theta_k) - v_m(\theta_k)| = 0.05$.

Note from Fig. 2 that the sidelobe level variance of delay-and-sum beamformer is much smaller than that of nominal optimal beamfomer, which means the former is much more robust in the presence of array manifold mismatch than the latter. The sidelobe level variance of the robust method is smaller than that of nominal optimal beamfomer and bigger than that of delay-and-sum beamformer. Hence, Fig. 1 and Fig. 2 show that our method provide another trade-off between the sidelobe level and the robustness.

Note from Fig. 1 and Fig. 2 that with uncertainty present, the sidelobe level of the nominal optimal beamformer increases rapidly, although its sidelobes are very low in the ideal case with no array manifold error. On the other hand, the sidelobe levels by our robust algorithm just degrade a little compared to the ideal case. Obviously, it provides sufficient robustness against array manifold errors as compared to the nominal optimal beamformer.

5 Conclusion

A robust array pattern synthesis approach based on the worst case performance optimization has been developed. The problem is reformulated as a convex optimization problem which can be solved efficiently using an SOCP solver. This beamformer can provide good robustness in the presence of array manifold perturbations. The advantage of the proposed technique is that the parameters can be optimally chosen based on the uncertainty level of the array manifold vector. The simple lower bound on how much worse the robust beamformer will be compared to the nominal one has also been derived. Results of numerical examples show good performance of the proposed approach in the case of uncertain array manifolds.

Acknowledgments

The author would like to thank Professor Jens M. Hovem and Professor Hefeng Dong at Norwegian University of Science and Technology for their discussions. This work was supported by the National Natural Science Foundation of China under Grant No. 60602055.

References

[1] H. L. Van Trees, *Optimum Array Processing: Part IV of Detection, Estimation, and Modulation Theory*. New York: John Wiley & Sons, Inc, 2002.

[2] J. Li, P. Stoica, *Robust Adaptive Beamforming*. New York: John Wiley & Sons, Inc, 2005.

[3] C. L. Dolph, "A current distribution for broadside arrays which optimizes the relationship between beam width and side-lobe level", *Proc. IRE* 34, 335-348 (1946)

[4] C. A. Olen, R. T. Compton, "A numerical pattern synthesis algorithm for arrays", *IEEE Trans. Antennas Propagat*. 38, 1666-1676 (1990)

[5] D. Dotlic, A. J. Zejak, "Arbitrary antenna array pattern synthesis using minimax algorithm", *Electronics Letters* 37, 206-208 (2001)

[6] H. Lebret, S. Boyd, "Antenna array pattern synthesis via convex optimization", *IEEE Trans. Signal Processing* 45, 526-532 (1997)

[7] J. Liu, A. B. Gershman, Z. Q. Luo, K. M. Wong, "Adaptive beamforming with sidelobe control: A second-order cone programming approach", *IEEE Signal Processing Lett*. 10, 331-334 (2003)

[8] S. F. Yan, C. H. Hou, X. C. Ma, Y. L. Ma, "Convex optimization based time-domain broadband beamforming with sidelobe control", *J. Acoust. Soc. Am*. 121, 46-49 (2007)

[9] S. F. Yan, Y. L. Ma, C. H. Hou, "Optimal array pattern synthesis for broadband arrays", *J. Acoust. Soc. Am*. 122, 2686-2696 (2007)

[10] H. Cox, R. M. Zeskind, M. M. Owen, "Robust adaptive beamforming," *IEEE Trans. Acoust. , Speech, Signal Processing* 35, 1365-1376 (1987)

[11] S. F. Yan, Y. L. Ma, "Robust supergain beamforming for circular array via second-order cone programming", *Applied Acoustics* 66, 1018-1032 (2005)

[12] B. D. Carlson, "Covariance-matrix estimation errors and diagonal loading in adaptive arrays", *IEEE Trans. Aerosp. Electron. Syst*. 24, 397-401 (1988)

[13] S. A. Vorobyov, A. B. Gershman, Z. Q. Luo, "Robust adaptive beamforming using worst-case performance optimization: A solution to the signal mismatch problem", *IEEE Trans. Signal Processing* 51, 313-324 (2003)

[14] P. Stoica, Z. S. Wang, J. Li, "Robust capon beamforming", *IEEE Signal Processing Lett*. 10, 172-175 (2003)

[15] J. Li, P. Stoica, Z. S. Wang, "On robust Capon beamforming and diagonal loading", *IEEE Trans. Signal Processing* 51, 1702-1715 (2003)

[16] J. Li, P. Stoica, Z. S. Wang, "Doubly constrained robust Capon beamformer", *IEEE Trans. Signal Processing* 52, 2407-2423 (2004)

[17] R. G. Lorenz, S. R. Boyd, "Robust minimum variance beamforming", *IEEE Trans. Signal Processing* 53, 1684-1696 (2005)

[18] Mutapcic, S. J. Kim, S. Boyd, "Beamforming with uncertain weights", *IEEE Signal Processing Lett*. 14, 348-351 (2007)

[19] J. F. Sturm, "Using SeDuMi 1.02, a MATLAB toolbox for optimization over symmetric cones", *Optim. Methods Software* 11-12, 625-653 (1999)

[20] M. S. Lobo, L. Vandenberghe, S. Boyd, H. Lebret, "Applications of second-order cone programming", *Linear Algebr. Appl*. 284, 193-228 (1998)