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The comparative analysis of two solutions of Pekeris boundary problem

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The incorrectness of boundary problems for the layered half-space in the classical statement is shown using an example of the Pekeris waveguide. The solution of the reduced Pekeris boundary problem satisfying the generalized radiation condition on an impedance interface is obtained. A mode part of the solution is presented by a series expansion on the total system of regular normal waves, generalized normal waves and leakage normal waves. The principle of the continuous continuation of the obtained solution in the physical half-space with validity of the boundedness condition is formulated. The expansion of the solution on the physical half-space in the class of generalized functions is constructed. Results of the comparative analysis of the classical solution and generalized solution are given. The existence of a subbottom wave at the water - sea bottom interface, which is dominating in the sound field of the ground projector, is theoretically predicted and experimentally confirmed within the limits of the generalized.

1 Introduction

The phenomenon of the total interior reflection plays a key role in the problems connected with the waveguide propagation of a sound. The first doubts about the completeness of the description of processes of reflection and refraction of sound waves using the classical theory at overcritical incidence angles arose in the 1970s. Several theories [1, 2] explaining the anomalous effects occurring at reflection and refraction of the sound beams at the overcritical incidence angles were proposed. One of them is the generalized theory offered by the authors.

As it was shown earlier [3], the Pekeris boundary problem in the initial statement [4] being a singular one by two independent coordinates (r, z) is not correct as eigen-functions of the cross operator in the complete set do not satisfy the disappearance condition and they are not bounded in the initial definition range. The correct statement of the Pekeris problem is possible only within the limits of the reduced boundary problem for a waveguide with the impedance boundary condition. However, in such formulation the admittance of the half-space is proportional to the cross wave number which is a two-place function of the complex spectral parameter.

For this reason and in conformity with the Maljuzhinets theorem of existence and uniqueness [5] the reduced boundary problem has a set of solutions. Each of them is unique in the corresponding modeling statement, which is unambiguously determined by the selection of a cut on a plane of the complex spectral parameter and the selection of the top (or physical) sheet. As shown with the help of analytical and numerical methods in the work [3], the known opinion about identity of these solutions stated in the work [6] is mistaken**.**

2 Fundamentals of the theory

Let us consider the problem of finding the field of the point source in the homogeneous liquid layer with depth h lying on the homogeneous liquid half-space. The sound speed in the layer is c_1, the sound speed in the half-space is c_2, the density of the layer is ρ_1, the density of the half-space is ρ_2. The classical solution of the Pekeris boundary problem corresponds to the following condition of assignment of a cut and a top sheet of the Riemann surface [6, 7, 8]

$$\operatorname{Im} k_{32} \le 0 \,, \ (L_1 \text{ cut}) \tag{1}$$

– cut of Ewing, Jardetzky and Press (EJP). Here $k_{32} = \sqrt{k_2^2 - \xi^2}$, $k_2 = \dfrac{\omega}{c_2}$, ω is angular frequency, ξ is the spectral parameter. The summary solution has the following canonical form:

$$\varphi^{(1)}(r, z) = \varphi_б^{(1)}(r, z) + \sum_{n(1)=1}^{N_1} \varphi_n(r, z, \xi_n), \tag{2}$$

where $\varphi_б^{(1)}(r, z)$ is an integral on coasts of the EJP cut referred to as a lateral wave, $\varphi_n(r, z, \xi_n)$ are regular normal waves corresponding to the real discrete spectrum of eigenvalues from the subset $n(1)$, N_1 is a number of normal waves, r, z are the horizontal coordinate and vertical coordinate respectively. The cross operator is self-conjugated in the given modeling statement, and the vertical power flow through the impedance interface is equal to zero both in the range of undercritical incident angles and in the range of overcritical incident angles. It means that the waveguide and the half-space do not have any energy connection between them.

Among the modelling solutions analysed by us earlier [3] only one solution contains the total set of normal waves (regular, generalized and leakage waves) corresponding the initial non-self-integrated operator and satisfies the generalized radiation condition which is given by

$$\operatorname{Re} k_{32} \ge 0 \,, \tag{3}$$

The condition (3) is a mathematical expression of the causality principle according to which power flow should be directed from the source to loading which role plays input admittance of the half-space (from cause to effect and not otherwise). In this case the general solution of the boundary problem for the potential $\varphi(r, z)$ is given by

$$\begin{aligned} \varphi(r, z) = \varphi_б(r, z) + \sum_{n(3)=1}^{\infty} \varphi_n(r, z, \xi_n) + \\ + \frac{1}{2}\left[\sum_{n(1)=1}^{N_1} \varphi_n(r, z, \xi_n) + \sum_{n(2)=1}^{N_2} \varphi_n(r, z, \xi_n) \right] \end{aligned}, \tag{4}$$

$\varphi_n(r, z, \xi_n)$ are the normal waves which are regular, generalized or leakage ones; $n(2)$ is a subset of generalized normal waves with real constants of propagation, N_2 is the number of generalized normal waves; $n(3)$ is a countable subset of leakage normal waves with the complex constants of propagation, $\varphi_{\delta}(r, z)$ is an integral on coasts of the cut determined by the condition (3).

The first generalized normal wave continues into the half-space as a inhomogeneous wave with the amplitude exponentially growing in the direction of an axis z up to the horizon of the total internal reflection $z_s = \rho_{12}(h - z_0)$ ($\rho_{12} = \rho_1 / \rho_2$, z_0 is depth of the source). The subbottom waveguide limited from below by the horizon of the total internal reflection as the caustic border is formed in the lower half-space. The interference interaction of the first pair of normal waves realizes cyclic leaking of the sound wave energy to the horizon of the total internal reflection followed by its return to the waveguide.

The presence of the conjugated normal waves (eigen-functions of conjugated operators) in the summary solution allows describing the process of the total internal reflection predicted by Newton in the range of overcritical incidence angles connected with leaking of energy to the horizon of the total internal reflection and its further return to the waveguide. The leakage normal waves allow describing adequately the near field of the projector in the waveguide.

This solution is most adapted for the description of the boundary wave processes with leaking of energy to a subbottom layer of the lower half-space down to the horizon of the total internal reflection with its further return to the waveguide, i.e. of the process of the energy interchange between two connected wave systems, such as the waveguide and the half-space.

The procedure of continuing the generalized solution into the physical half-space is connected with its additional correction, which is reduced to finding the diffraction corrections providing continuity of the field by pressure and the normal component of particle velocity on the interface. However, the correct solution satisfying the condition of disappearance at infinity can be constructed only in the class of the generalized functions which is researched in detail in the work [3].

3 Numerical modeling results

The numerical analysis of modeling solutions has been executed for the Pekeris waveguide with parameters $\rho_{12} = \rho_1 / \rho_2 = 1/1.6$; $c_{12} = c_1/c_2 = 1.5/1.75$ for the frequency parameter $k_1 h = 200$ and different radiation horizons $z_0 / h = z_{01}$, which are normalized by the depth of the waveguide h, $z_{s1} = z_s / h$. The value $\hat{\varphi}(r) = \varphi(r)\sqrt{r_1}$; $r_1 = r/h$ with the excluded cylindrical divergence is taken as the design value. Results of the calculation of the sound field corresponding to the generalized solution are shown in Fig.1. The sound field corresponding to the EJP cut (classical solution) is shown in Fig. 2.

The compared sound fields have the fundamental differences in the near-field zone and in the boundary zone. In the near-field zone of the projector, which is limited by a beam falling on the interface under a critical angle, the sound field calculated according to the generalized theory is described by the complete set of the converging and diverging leakage normal waves. Features of the near-field zone of the projector are connected with the special character of leaking and the characteristic focusing of radiation near the symmetry axis.

The secondary projector of the piston type is formed on the interface within the limits of the near-field zone. On the projector surface pressure and the normal component of particle velocity have the in-phase components responsible for power radiation into the lower half-space. This secondary projector, in its turn, forms a field in the half-space with clearly defined elements of the field focusing on the symmetry axis. Both diverging waves and converging waves, which become diverging only after their reflection from the symmetry axis as from the rigid boundary, take part in this focusing. The focusing effect manifests itself in the appearance of real focusing points with coordinates $z_f = 2h \pm z_0$ and in the appearance of the caustic surface which is a bending envelope of converging leakage waves (beams). Formation of a caustic can be seen in Fig. 1.

As a result the field is formed in the half-space mainly by leakage waves each of which leaks to the corridor of leaking producing the focusing zone on the axis of symmetry, which are characteristic of the piston-type projector. The angular structure of the field completely corresponds to the complex spectrum of leakage normal waves, and the flow of power through the interface is non-zero.

Another distinctive feature of the solution is the presence of the boundary sound channel and the subbottom wave localized in it, which realize the Newton hypothesis about the total internal reflection. This component is dominating in the summary sound field localized near the water – sea bottom interface when the projector is located near the bottom (Fig. 1*b*). In Figures 1*a* and 1*b* you can see the lower border of the boundary sound channel in the form of a bright white line and also the change in depth of the boundary sound channel depending on the height of the source above the bottom.

The classical solution corresponding to the self-conjugated modeling statement differs in principle from the generalized solution. The secondary projectors which have appeared on the axis of symmetry in consequence of the accepted solution model completely compensate the real effect of focusing and reduce the real field near the symmetry axis. As a result of such a “defocusing” effect the local minimum of sound pressure is formed on the axis of symmetry. At the same time the flow of power through the interface is equal to zero at all incidence angles. It is a characteristic feature of the solution to the problem in the self-conjugate modeling statement. The field attenuation near the axis of symmetry can be seen in Fig. 2. Another characteristic feature is the absence of any attributes of the total internal reflection with the energy leaking to the corresponding horizon of the total internal reflection and its further return to the waveguide.

In contrast to the classical solution the obtained solution is generalized based on the class of the functions used for its construction, and it can form a basis of the generalized theory of normal waves in the layered waveguides loaded on the physical half-space. Its distinctive feature is the appearance of the subbottom wave on an impedance interface which contains three partial components: the first pair of normal waves, a regular one and a generalized one, and a lateral wave. The beam treatment of the subbottom wave generation is presented in Fig. 3.

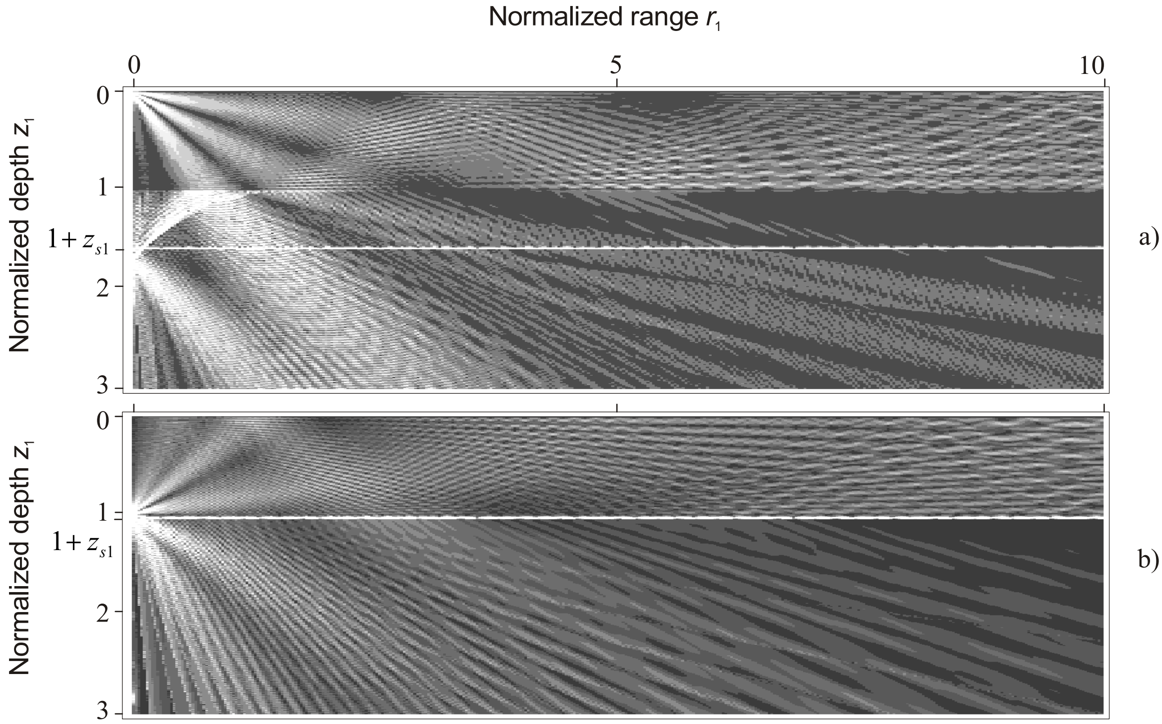

Fig.1 The sound field in the waveguide and half-space corresponding to the generalized solution, $k_1h = 200$: a) $z_{01} = 0.05$; b) $z_{01} = 0.95$.

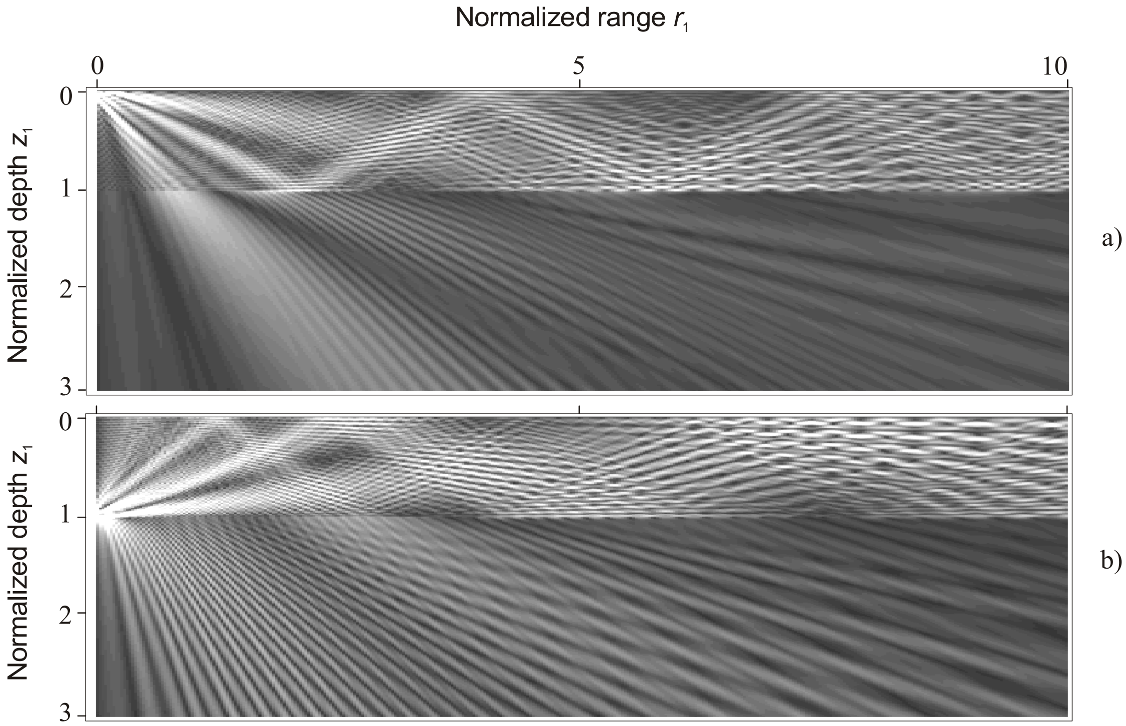

Fig.2 The sound field in the waveguide and half-space corresponding to the classical solution (the EJP cut), $k_1h = 200$: a) $z_{01} = 0.05$; б) $z_{01} = 0.95$.

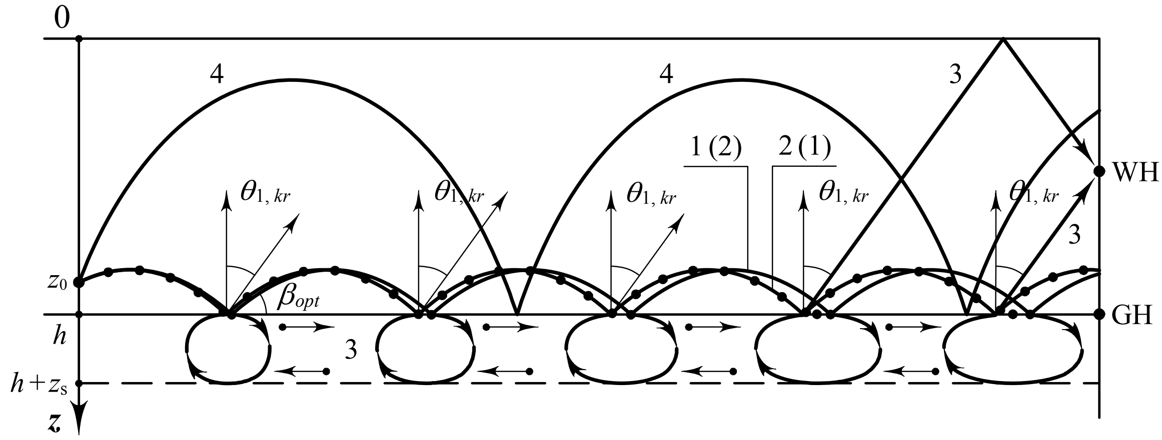

Fig.3 The beam treatment of the subbottom wave, 1(2) - the first generalized subbottom wave, 2(1) - the second regular normal wave, 3 – the lateral (ground) wave, 4 – the high-velocity beam of the subbottom sound channel, WH, GH - water and ground hydrophones.

The velocity of the subbottom wave propagation does not depend on frequency and is close to sound velocity in water as with all boundary waves. It means that the effect of the total internal reflection compensates the effect of the soft screen and enables the propagation of the subbottom wave at arbitrarily small grazing angles.

4 Experimental results

In order to confirm experimentally the fact of existence of the subbottom wave and to define its kinematical characteristics a series of experiments were organized in the frequency band of 369-2526 Hz in the shallow sea with the depth of 40-60 м at the hydrophysical range of the Pacific Oceanological Institute (POI FEB RAS) in Peter the Great bay (the Sea of Japan) [9] in 2005 - 2007.

The directional projector was placed at the bottom by means of a ground anchor. The second carrier ship anchored at the given point of the path and lowered a two-channel receiving system down to the bottom. Coordinates of the carrier ships were measured continuously by means of GPS-receivers at the mounting points. The group velocity of propagation of an acoustic signal was calculated by the measured distance and propagation time.

Results of all the experiments carried out in 2005-2007 showed good conformity both to each other and the generalized theory. The results of the comparative estimation of propagation velocity of a fast mode of the refraction type (the ray theory), a subbottom wave (the generalized theory) and group velocity of propagation of the signal measured in all three experiments are presented in Fig. 4 – Fig. 6. We can note the presence of some transition zone (a near zone) in the data of the three experiments, in which the propagation velocity of a signal differs from the one predicted by the ray theory or generalized theory. However, as the distance grows, the velocity of the signal propagation tends steadily to the low-velocity limit predicted by the generalized theory. At this low-velocity limit the velocity of the signal propagation is $1-3$ m/c lower than the sound velocity near the bottom.

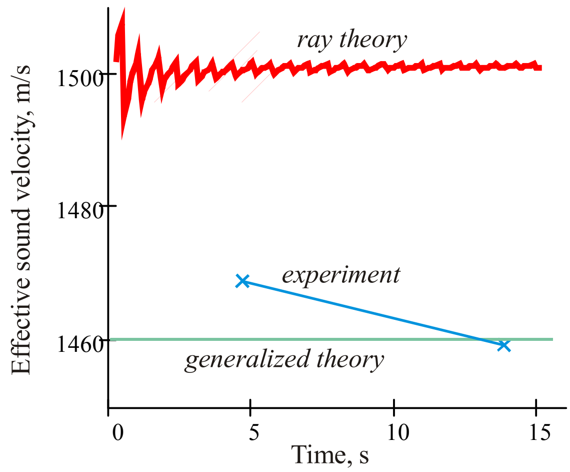

Fig.4 An example of the comparative estimation of the beam velocity (the ray theory), velocities of a subbottom wave (the generalized theory) and the group velocity of a signal (experiment) in the autumn experiment of 2005 on the 20-km path in the frequency band of 369-2526 Hz.

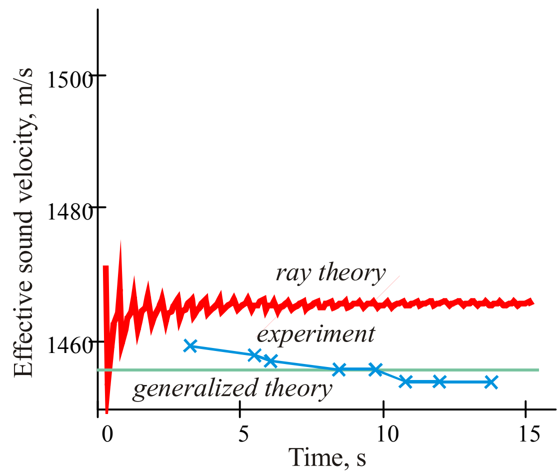

Fig.5 An example of the comparative estimation of the beam velocity (the ray theory), velocities of a subbottom wave (the generalized theory) and the group velocity of a signal (experiment) the spring experiment of 2006 on the 20-km path in the frequency band of 369-2526 Hz.

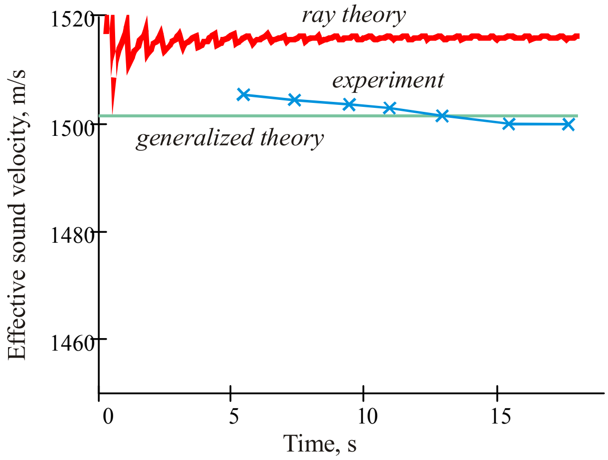

Fig.6 An example of the comparative estimation of the beam velocity (the ray theory), velocities of a subbottom wave (the generalized theory) and the group velocity of a signal (experiment) for the autumn of 2007 experiment on the 27-km path at the frequency of 1.9 kHz.

5 Conclusion

The main results of the research performed are briefly formulated below.

The correct formulation of the boundary problem corresponding to the Pekeris problem is possible only within the limits of the reduced boundary problem for the waveguide with a boundary condition of the mixed type (the impedance type). The reduced boundary problem has a set of solutions; each of them is unique in the corresponding modeling statement.

The generalized solution contains the complete set of normal waves, i.e. regular, generalized and leakage waves, and it allows describing the process of the total internal reflection predicted by Newton in the range of overcritical incidence angles connected with leaking of energy to the horizon of the total internal reflection and its further return to the waveguide. According to the generalized solution the subbottom waveguide, with the subbottom wave localized in it, is formed in the lower half-space.

The velocity of propagation of the subbottom wave determined on the basis of the experimental research corresponds well to the minimal value of sound velocity in water near the bottom; it depends loosely on the hydrology acoustic conditions of propagation and has no frequency dependence.

Acknowledgments

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