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New approach to the problem on long-wave sound scattering by a Rankine vortex

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The problem of sound scattering by a Rankine vortex at small Mach number is considered. Despite the long history of the problem, solutions obtained by different authors still are not free from essential objections. The main difficulty consists in that the slow decay of the mean velocity field at infinity hinders the correct formulation of the problem. Most authors use a plane wave as the incident field, which is a solution of the governing equations in the leading approximation only. However, this field cannot unambiguously provide the second approximation, which is needed to determine the scattering field. The matter is that there are many incident waves which correspond to plane wave in the main approximation but differ in the second one and only physical statement is defined it. We suggest instead of the plane wave condition to pose the condition of a point source at large but finite distance from the vortex; it allows us to determine unambiguously the incident field to any approximation in Mach number. In the new formulation a correct solution of the problem of non-resonant scattering is obtained. Existing solutions for resonant scattering are analysed too and a result unifying the previous ones is found.

1 Introduction

The problem of sound scattering by a two-dimensional circular vortex has received significant attention, because it is a basic problem of the theory of sound interaction with hydrodynamic flows. Three main approaches to deal with it can be distinguished.

The first one is based on a solution for Lighthill's equation with the source defined by the plane wave [1,2,3]. The expression for the scattering amplitude obtained in these papers is given by

$$f(\theta) \sim M^2 \left(\sin \theta - \operatorname{ctg} \frac{\theta}{2} \right) \quad (1)$$

Here θ is the scattering angle; M is the Mach number, which is assumed to be small. Solutions of the problem without the singularity at $\theta=0$ are obtained in [4,5].

The second approach is based on a solution for Howe's equation for the stagnation enthalpy [6,7]. The result obtained in [7] also possesses no singularity. The third approach to the scattering problem is based on the partial harmonic expansion of the plane wave [8,9]. Attempts to calculate the scattering field numerically have also been made [10,11,12].

However, it appears (see [13]) that the abovementioned solutions are not free from essential objections. In all these papers the incident sound field is assumed to be a plane wave. The difficulties result from the fact that at small Mach numbers the plane wave condition can unambiguously provide only the leading-order term of the expansion of the incident field in Mach number (this corresponds to the situation when the total vorticity is zero), while to determine the scattering field one must know the higher-order terms in the expansion. Therefore, it is desirable to use the problem formulation that (i) is physically reasonable and (ii) unambiguously provides the higher-order terms in the expansion of the incident field in Mach number.

Therefore, we propose to consider as the incident field the field of a point harmonic source placed at far but finite distance from the vortex. The point source model satisfies both abovementioned conditions. It should be noted that a point source model is also used in [2]. However, the main attention in that paper is paid to the case of a point source being *near* the vortex. The case of the point source being far from the vortex is, by analogy with the usual diffraction problems, substituted by a problem of plane wave

scattering, i.e. the case of a point source at large but finite distance from the vortex has not, in fact, been investigated.

There is also an important question about resonant scattering by the vortex. In [9] it is demonstrated that when the incident frequency coincides with a resonant frequency (i.e. the real part of an eigen-frequency) of the vortex, the scattering amplitude becomes of $O(1)$, not $O(M^2)$ as for the case of non-resonant scattering. However, in [14] it is proved by using matched asymptotic expansions to $O(M^2)$ that although the resonant scattering amplitude increases, it does not attain to $O(1)$. In §4 we demonstrate that both these results are correct. The resonant scattering amplitude may indeed be of order unity as is obtained in [9]. However, this occurs not at the incompressible vortex frequency, but at the exact compressible vortex frequency, which differs from the incompressible vortex frequency by terms of $O(M^4)$. If the incident frequency coincides with the eigen-frequency of the incompressible problem, then the real part in the denominator of the scattering amplitude does not vanish and one obtains the result of [14].

Thus, in §2 the governing equations are considered. In §3 non-resonant sound scattering by the Rankine vortex in the weakly compressible fluid approximation is investigated in the new formulation. In §4 the result for resonant scattering that unifies the previous ones and resolves the contradictions between them is obtained.

2 Governing equations

Let there be a Rankine vortex of radius a in a perfect (inviscid and non-heat-conducting) compressible fluid. In the cylindrical coordinate system (r, θ, z) it means that inside the circle of radius $r = a$ centered at the point of origin, the z -component of vorticity is a constant Ω_0 and the other components are zero. There is no vorticity outside the circle. The flow is supposed to be isentropic and independent of z .

The propagation of the acoustic disturbances with the frequency ω_0 is governed by the linearized Euler equations (LEE). The solution for the LEE will be sought as a harmonic series:

$$p(r, \theta, t) = e^{-i\omega_0 t} \sum_{n=-\infty}^{+\infty} p_n(r) e^{in\theta} \quad (2)$$

for the pressure and in a similar manner for the density and the components of velocity. That is, we follow the third (partial harmonic expansion) approach to solve the problem.

We will employ the dimensionless variables that are made such by using the values of the corresponding mean field variables at the vortex boundary ($r=a$), e.g.

$$x = \frac{r}{a}, \quad \omega = \frac{\omega_0}{\Omega_0/2}, \quad M = \frac{\Omega_0 a/2}{a_0(\alpha)} \quad (3)$$

etc. Here $a_0(\alpha)$ is the sound speed at the vortex boundary.

In the region $x < 1$ (i.e., inside the vortex) the LEE can be reduced, when the $O(M^4)$ terms are neglected, to the following equation for the n th pressure harmonic:

$$\frac{d^2 p_n}{dx^2} + \frac{1}{x}(1 - M^2 x^2) \frac{dp_n}{dx} + \left[M^2 \Psi - \frac{n^2}{x^2} \right] p_n = 0 \quad (4)$$

Here $\Psi = (n - \omega)^2 - 6 + \frac{2n}{n - \omega} \equiv \text{const.}$ When p_n is

known, the LEE enables us to determine the other variables straightforwardly.

In the region $x > 1$ (i.e., outside the vortex) we may introduce the velocity potential ϕ . The LEE can be reduced to the following equations for the n th velocity potential harmonic ϕ_n : in the region near the boundary vortex ($x \sim 1$), when the $O(M^4)$ terms are neglected, we obtain

$$\begin{aligned} \frac{\partial^2 \phi_n}{\partial x^2} + \frac{1}{x} \left(1 + \frac{M^2}{x^2} \right) \frac{\partial \phi_n}{\partial x} + \\ \left(M^2 \omega^2 + M^2 \frac{n^2}{x^4} - \frac{v_0^2}{x^2} \right) \phi_n = 0 \end{aligned} \quad (5)$$

Here $v_0^2 = n^2 + 2n\omega M^2$. In the region far away from the vortex ($x \gg 1$) the equation is as follows:

$$\frac{d^2 \phi_n}{d\xi^2} + \frac{1}{\xi} \frac{d\phi_n}{d\xi} + \left(1 - \frac{|v|^2}{\xi^2} \right) \phi_n = 0 \quad (6)$$

Here $v^2 = n^2 + 2n\omega M^2 + \omega^2 M^4$; $\xi = kr = Mx$. When ϕ_n is known, the LEE enables us to determine the other variables straightforwardly.

3 Non-resonant scattering

Let there be a point source of sound (a mass source) at the point $\xi = kR$, $\theta = \pi$. It means that in the right-hand side of Eq.(6) a source term appears, and the equation takes on the form

$$\frac{d^2 \phi_n}{d\xi^2} + \frac{1}{\xi} \frac{d\phi_n}{d\xi} + \left(1 - \frac{|v|^2}{\xi^2} \right) \phi_n = q \frac{e^{-i\pi n}}{2\pi R} \delta(\xi - kR) \quad (7)$$

Here q is the source strength, which from now on is set to be 1. The solution of this equation in the region $\xi < kR$ is as follows:

$$\phi_n = a_n J_{|v|}(\xi) + F_n H_{|v|}^{(1)}(\xi) \quad (8)$$

Here $a_n = -ie^{-i\pi n} H_{|v|}^{(1)}(kR)/4$, $J_n(\xi)$ is the Bessel function and $H_n(\xi)$ is the Hankel function.

The first term in Eq.(8) corresponds to the incident waves; the second term corresponds to the outgoing (scattered) waves. Thus, to determine the scattering field, it is necessary to determine the amplitudes F_n for all n . This can be done as follows. First of all, we obtain the solutions of Eq.(4) and Eq.(5) with the accuracy to the $O(M^4)$ terms. Each of these solutions contains two unknown constants. Three of these four constants are determined via the requirements that the solutions must be finite at $r=0$ and that the pressure and the radial component of velocity must be continuous at the vortex boundary. This leaves only one constant, say C , unknown. Note furthermore that as the coordinate x increases, the solution of Eq.(5) with the unknown constant C must transform into Eq.(8) with the unknown constant F_n . Van Dyke's matching principle enables us to perform the matching of these solutions and thus to determine both C and F_n . The tedious calculations provide the following expression for F_n :

$$F_n = \begin{cases} -\pi\omega M^2 \text{sgn}(n) H_{|v|}^{(1)}(\xi)/16, & n = \pm 1 \\ O(M^4), & n \neq \pm 1 \end{cases} \quad (9)$$

Thus, in the region $k\alpha \ll \xi < kR$ the sound field is as follows:

$$\phi = \Phi - \frac{i\pi}{8} \omega M^2 H_1^{(1)}(kR) H_1^{(1)}(kr) e^{-i\omega_0 t} \sin\theta \quad (10)$$

$$\Phi = \sum_{n=-\infty}^{+\infty} \frac{e^{-i\pi n}}{4i} H_{|v|}^{(1)}(kR) J_{|v|}(kr) e^{in\theta - i\omega_0 t} \quad (11)$$

The second term in Eq.(10) constitutes the proper scattering field and has been correctly obtained by all researchers. It is the first term that caused the most trouble. This term is due to the refraction of sound on the slowly decaying mean velocity field, and there has been quite a controversy about how it must be calculated. The plane wave condition results in the necessity to compute multiple integrals that are not correctly defined. Different authors transform the multiple integrals into different iterated integrals and therefore obtain different, contradicting expressions for the refracted field Φ . A more detailed discussion of this can be found in [13]. The point source model delivers us from arising of such integrals and leads to the expression for the refracted field given by Eq.(11).

It turns out that the sum of the series in Eq.(11) can be analytically determined in the region $k\alpha \ll \xi \ll kR$ [13]. The Bessel and Hankel functions are exponentially small when v is less than the argument of the function; therefore, the Hankel function in the series may be replaced by its large argument asymptotic form:

$$\Phi = \sum_{n=-\infty}^{+\infty} \frac{e^{-i\pi n}}{4i} J_{|v|}(kr) \sqrt{\frac{2}{\pi kR}} e^{i\left(kR - \frac{\pi|v|}{2} - \frac{\pi}{4}\right)} e^{in\theta - i\omega_0 t} \quad (12)$$

Then we use the Schl\"{a}fli integral representation for the Bessel function, change the order of summation and integration, calculate the geometric progression and finally perform the integration by making use of a saddle-point method. As a result, we obtain that in case θ is not close to 0 , π or 2π

$$\Phi = Ke^{i\omega M^2(\pi-\theta)} e^{ikr\cos\theta} + \frac{K\omega M^2}{\sqrt{2\pi kr}} \pi \operatorname{ctg} \left(\frac{\theta}{2} \right) e^{ikr - i\frac{\pi}{4}} \quad (13)$$

$$K = \frac{e^{-i\omega t}}{4i} \sqrt{\frac{2}{\pi kr}} e^{i\left(kR - \frac{\pi}{4}\right)} \quad (14)$$

The first term in Eq.(13) is a quasi-plane wave, incident on the Rankine vortex. The second term constitutes a refracted outgoing cylindrical wave. Outside the considered region $k\alpha \ll \xi \ll kR$ the original expression (see Eq.(11)) must be used.

In case θ is close to $0, \pi$ or 2π , a pole of the integrand of the Schlöffli integral becomes close to a saddle point. The situation can be dealt with analytically and gives a cumbersome expression via the error integrals, resulting in the absence of singularities at these angles, as has already been obtained in [4,5,7].

4 Resonant scattering

The Rankine vortex is known to be an oscillatory system that can emit sound in a compressible fluid [9,15]. Sound scattering may cause resonant excitation of oscillations and the scattering field (re-emission by excited degrees of freedom) may increase by many times. In this case, eigen-frequencies of the system become poles of the scattering amplitude. Since the eigen-frequencies of the emitting vortex have imaginary parts (that correspond to the instability – see [9,15]), the denominator of the scattering amplitude does not vanish identically at the real-valued incident frequency, and the amplitude is given by

$$F_n \sim \frac{i\delta_n}{\omega - \omega_n - i\delta_n} \quad (15)$$

Here ω is the dimensionless incident frequency; ω_n and δ_n are, correspondingly, the real and imaginary parts of the dimensionless eigen-frequency. Eq.(15) is obtained in [9], where $\delta_n \sim M^{2n}$ is an increment for each unstable harmonic, calculated for the first time for $n=2$ in [15]. This structure of the solution demonstrates that the amplitude of the stationary solution is finite; the situation is analogous to scattering by quasi-discrete levels of energy in quantum mechanics. The resonance scattering is possible only for $|n| \geq 2$ (eigen-oscillations of the vortex exist for these n).

Non-resonant scattering for these harmonics is small (it is of $O(M^4)$ or smaller) and can be neglected in comparison with the first mode. As it follows from Eq.(15), when the incident frequency coincides with ω_n the scattering amplitude is of order unity, which is larger than its non-resonant contribution (9). In [14] this conclusion is meticulously analyzed with the accuracy to $O(M^2)$ and it is shown that when the incident frequency ω coincides with the eigen-frequency of the incompressible Rankine vortex $\omega_n = n-1$, the scattering amplitude is smaller than unity; for example, in the case of $n=2$ it increases only to the value $\sim M^2$, and not to the unity. This, at the first sight, renders the resonant scattering inefficient. However, it turns out that the strong resonant scattering does indeed take place; the resonance occurs not at the incompressible vortex eigen-frequency, but at the compressible vortex eigen-frequency. Let us consider the problem in more detail.

Let us restrict ourselves to the case $n=2$. To determine the scattering amplitude, the $O(M^4)$ terms are necessary; therefore, we must derive from the LEE the equations, analogous to Eqs.(4)–(6) but valid with the accuracy to $O(M^4)$. To determine the compressible vortex eigen-frequency, the coefficient a_2 in Eq.(8) must be set equal to zero, i.e. we have to consider the case when only the outgoing waves are present. Performing the asymptotic matching, analogous to the non-resonant case, we obtain for this frequency:

$$\omega_2 = 1 - \frac{M^2}{12} - M^4 \left(\frac{67}{1152} + \frac{C_\gamma}{16} - \frac{i\pi}{32} + \frac{\gamma}{192} + \frac{1}{16} \ln \frac{M}{2} \right) \quad (16)$$

Here $C_\gamma = 0.5772\dots$ is the Euler constant and γ is the heat capacity ratio. This expression coincides with that of [15]; it should be noted, however, that there seems to be a misprint in [15]: the coefficient in the first term in the parentheses must be $67/1152$, not $67/1162$.

Calculation of the scattering amplitude for the sound wave with the frequency ω close to the resonant frequency ω_2 gives

$$F_2 = \frac{a_2 \frac{i\pi}{32}}{\frac{67}{1152} + \frac{C_\gamma}{16} + \frac{\gamma}{192} + \frac{1}{16} \ln \left(\frac{M}{2} \right) - w - \frac{i\pi}{32}} \quad (17)$$

$$\omega = 1 - \frac{M^2}{12} - wM^4 \quad (18)$$

It is manifest that the resonance scattering amplitude in Eq.(17) is indeed of $O(1)$ and not of $O(M^2)$, and has the structure exactly described by (15) (as it has been predicted in [9]). However, the resonance, obviously, occurs at the frequency close to that of Eq.(16), i.e. the compressible vortex eigen-frequency. If the incident frequency ω coincides with an eigen-frequency of the incompressible vortex (i.e., coincides with that of Eq.(16) in the leading-order terms), the largest term in the denominator of Eq.(15) is $\omega - \omega_2 \sim M^2$, and the numerator $\delta_2 \sim M^4$ does not cancel. Therefore, the strong resonance does not occur at the incompressible eigen-frequency and the scattering amplitude is, obviously, of $O(M^2)$, as has been predicted in [14].

5 Conclusion

The paper is dedicated to investigation of the well-known problem of long-wave sound scattering by the Rankine vortex. The problem is basic for investigation of the sound-vortices interaction, but there was some confusion so far. A new formulation is proposed, which is physically and mathematically correct; namely, sound scattering from a point harmonic source placed at large but finite distance from the vortex.

In the new formulation, the exact solution of the problem of non-resonant sound scattering by the Rankine vortex in the weakly compressible approximation is obtained. The

scattering field in the region outside the vortex does not possess a singularity at any scattering angle.

Resonant scattering is considered in the new formulation (with the point source) and the exact solution is obtained, which unifies the previous results and resolves the existing contradiction between them.

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