Intermodal coupling in a dissipative fluid filling a rough-walled waveguide

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The present study follows recent works dealing with the analytical model of an acoustic field in fluid-filled waveguides with rough walls. In these works, the acoustic field is obtained from the coupling between Neumann eigenmodes of the regularly shaped waveguide which bounds outwardly the rough walls of the waveguide considered, using integral formulation with suitable Green function. The effect of the roughness is expressed in such a way that two intermodal coupling mechanisms are highlighted: a bulk coupling and a surface coupling, the first one depending on the depth of the roughness and the second one depending in addition on the local slope. Moreover, a phonon relation is involved when the rough profile is periodic. The aim of the present study is to account for the thermo-viscous boundary layer effects through eigenmodes which satisfy appropriate mixed boundary conditions, leading to a better understanding of the physical mechanisms when resonances and phonon relationship are involved.

1 Introduction

Rough surfaces are of industrial interest, in the aim of improving the wetting of the glue in bounded structures for example. Guided waves like Lamb waves in solid structures are very useful to control such structures and may give some information on the roughness of the surface then on the quality of the wetting [1-3]. Former theoretical and experimental studies have permitted to bring to the fore a decay of a Lamb mode, with a probable energy transfer between modes [4-6].

Recent works [7] provide a first analytical approach of an acoustic field in fluid filled waveguides with rough wall. This approach departs from these available until now because it does not make use of the so-called multi-modal approach. In these works, the acoustic field is obtained from the coupling between Neumann eigenmodes of the regularly shaped waveguide which bounds outwardly the rough walls of the waveguide considered, using integral formulation with suitable Green function. The effect of the roughness is expressed in such a way that two intermodal coupling mechanisms are highlighted: a bulk coupling and a surface coupling, the first one depending on the depth of the roughness and the second one depending in addition on the local slope. Moreover, a phonon relation is involved when the rough profile is periodic.

The aim of the present study is to account for the thermo-viscous boundary layer effects through eigenmodes which satisfy appropriate mixed boundary conditions, leading to a better understanding of the physical mechanisms when resonances and phonon relationship are involved.

2 The fundamental problem

The fluid-filled waveguide considered here is assumed to be limited by two parallel rigid plates having two dimensional shape perturbations (three-dimensional problem). The fluid plate with rigid, regularly shaped surfaces $z=0$ and $z=\delta$, which encloses the real waveguide, is characterized by its thickness δ, the inner plate surrounded by the real waveguide is characterized by its thickness d (see Fig .1). The depth of the small shape deviations are respectively denoted $h_1(x,y)$ and $h_2(x,y)$ at $z_1=h_1$ and $z_2=\delta-h_2$.

In order to account for the dissipation phenomena both to provide a better approach of the physical mechanisms when resonances and phonon relationship are involved and to show the relative importance of the dissipation phenomena compared to those linked to the diffusion to the roughness, thermo-viscous boundary layer effects are accounted for through eigenmodes which satisfy appropriate mixed boundary conditions.

Therefore, the fluid is characterized by its density ρ_0, the adiabatic speed of sound c_0, its shear viscosity coefficient μ_0, its thermal conductivity coefficient λ_0 and its specific heat ratio γ, and its heat capacity at constant pressure per unit volume C_P.

The motion is supposed harmonic with ω the angular frequency (the time dependance being $\exp(i\omega t)$).

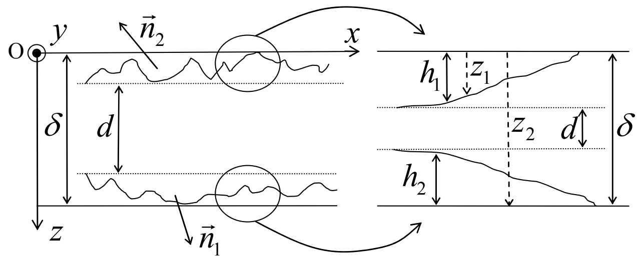


Fig.1 Geometry of the fluid-filled waveguide.

2.1 The dissipation phenomena

Considering the interior problem mentionned above, losses must be taken into account when the frequency is monitored to make the field resonant (or one or several modes resonant) because the solutions must behave accurately, and therefore it is important to express adequately the dissipation processes, namely the viscous and thermal effects in the boundary layers near the rigid walls. In the problems considered herein, explicit representations of the acoustic field, taking into account the effect of viscosity and thermal conduction, implies that several operations and approximations are carried out in the derivation of the appropriate basic equations involved as well as in their solutions. Then, the thermo-viscous boundary layer effects are taken into account through eigenmodes, which satisfy appropriate mixed boundary conditions with the appropriate admittance $\hat{Y}$ which takes the form [8].

$$\hat{Y}=\frac{(1+i)}{\sqrt{2}}\sqrt{k_0}\left[\sqrt{\frac{\mu_0}{\rho_0 c_0}}+(\gamma-1)\sqrt{\frac{\lambda_0}{\rho_0 c_0 C_P}}\right], \tag{1}$$

the adiabatic wavenumber k_0 being defined by $k_0=\omega/c_0$. It is worth noting that the dissipative processes in the bulk of the fluid can be modelled by substituting the wellknown appropriate complexe wave numbers to the wave number k_0.

2.2 The fundamental problem

The boundary conditions satisfied by the acoustic field on the perturbed surface of the waveguide are given by the requirement that its normal derivative on the surface vanishes at every point of the boundary. Denoting the local unit vectors $\vec{n}_1$ and $\vec{n}_2$ normal to the real surfaces of the waveguide respectively at the point z_1 and z_2 and pointing outside the fluid, the normal derivatives takes the classical form

$$\begin{cases} \partial_{n_1} = \dfrac{1}{N_1}\left[\left(\partial_x h_1\right)\partial_x + \left(\partial_y h_1\right)\partial_y - \partial_z\right], & (2.a) \\ \partial_{n_2} = \dfrac{1}{N_2}\left[\left(\partial_x h_2\right)\partial_x + \left(\partial_y h_2\right)\partial_y + \partial_z\right], & (2.b) \end{cases}$$

with

$$\begin{cases} N_1 = \sqrt{\left(\partial_x h_1\right)^2 + \left(\partial_y h_1\right)^2 + 1}, & (3.a) \\ N_2 = \sqrt{\left(\partial_x h_2\right)^2 + \left(\partial_y h_2\right)^2 + 1}. & (3.b) \end{cases}$$

The acoustic pressure field is governed by the set of equations including the propagation equation and mixed boundary conditions, which takes the following form

$$\begin{cases} \left(\Delta + k_0^2\right)\hat{p}(x,y,z) = -\hat{f}(x,y,z), \forall(x,y), z \in \left[z_1, z_2\right], & (4.a) \\ \partial_z \hat{p}(x,y,z) = O(x,y,z)\hat{p}(x,y,z), \forall(x,y), \text{at } z = z_1, & (4.b) \\ \partial_z \hat{p}(x,y,z) = O(x,y,z)\hat{p}(x,y,z), \forall(x,y), \text{at } z = z_2, & (4.c) \end{cases}$$

where $\hat{p}$ is the complex acoustic pressure and $\hat{f}$ the bulk source factor and where the operator $O(x,y,z)$ is defined by

$$O(x,y,z) = \left\{\left[(-1)^{q+1}\left(\partial_x h_q\right)\partial_x + \left(\partial_y h_q\right)\partial_y\right] + ik_0 Y N_q\right\}, \ \forall x, y,\ z = z_q,\ q = 1,2. \quad (5)$$

2.3 Modal wave decomposition

In order to model the sound absorption in the thermo-viscous boundary layers, the eigenvalue problem, associated to the problem Eq. (4), includes the mixed boundary condition which involve the admittance $\hat{Y}$ Eq. (1). It is expressed as a 1-D transverse (in the z-direction) eigenvalue problem, namely

$$\begin{cases} \left(\partial_{zz}^2 + \hat{\chi}_m^2\right)\hat{\psi}_m(z) = 0, \quad z \in [0, \delta], & (6.a) \\ \left(\partial_z - ik_0\hat{Y}\right)\hat{\psi}_m(z) = 0, \quad \text{at } z = 0, & (6.b) \\ \left(\partial_z + ik_0\hat{Y}\right)\hat{\psi}_m(z) = 0, \quad \text{at } z = \delta, & (6.c) \end{cases}$$

where the eigenvalues $\hat{\chi}_m$ are given by

$$\begin{cases} \hat{\chi}_0 \approx \sqrt{\dfrac{i2k_0\hat{Y}}{\delta}}, \ \text{if } m = 0, & (7.a) \\ \hat{\chi}_m \approx k_m + \dfrac{i2k_0\hat{Y}}{k_m\delta} = 0, \ \text{if } m \neq 0, & (7.b) \end{cases}$$

$k_m = m\pi/\delta$ being the eigenvalues for the problem with Neumann condition at the boundaries, and where the eigenfunctions (normalised and orthogonal) $\hat{\psi}_m$ are given by [9]

$$\hat{\psi}_m(z) = \frac{1}{\hat{C}_m}\left[\ \hat{\chi}_m \cos\left(\hat{\chi}_m z\right) + ik_0\hat{Y}\sin\left(\hat{\chi}_m z\right)\ \right], \quad (8)$$

with

$$\begin{cases} \hat{C}_0 \approx \sqrt{2ik_0\hat{Y}}, \ \text{if } m = 0, & (9.a) \\ \hat{C}_m \approx \hat{\chi}_m\sqrt{\delta/2}, \ \text{if } m \neq 0. & (9.b) \end{cases}$$

Expanding the pressure field $\hat{p}$ on the eigenfunctions $\hat{\psi}_m$,

$$\hat{p}(x,y,z) = \sum_\mu \hat{A}_\mu(x,y)\hat{\psi}_\mu(z), \quad (10)$$

solution of the posed problem for the acoustic pressure field takes the following form, with the help of the orthogonal properties of the eigenfunctions,

$$\begin{aligned} \left(\Delta_{//} + \hat{k}_{x_m}^2\right)\hat{A}_m(x,y) &= -\hat{S}_m(x,y) \\ &+ \sum_\mu \hat{\gamma}_{\mu m}(x,y)\hat{A}_\mu(x,y) \\ &+ \sum_\mu \hat{b}_{\mu m}(x,y)\left(\Delta_{//} + \hat{k}_{x_m}^2\right)\hat{A}_\mu(x,y), \quad (11) \end{aligned}$$

with

$$\Delta_{//} = \partial_{xx}^2 + \partial_{yy}^2, \quad (12)$$

$$\hat{k}_{x_m}^2 = k_0^2 - \hat{\chi}_m^2, \quad (13)$$

where the source term takes the form

$$\hat{S}_m(x,y) = \int_{z_1}^{z_2} \hat{\psi}_m(z)\hat{f}(x,y,z)\,\mathrm{d}z, \quad (14)$$

and where the coupling factors take the following expressions

$$\hat{\gamma}_{\mu m}(x,y)=\left[\hat{\psi}_m(z)O(x,y,z)\hat{\psi}_\mu(z)\right.$$

$$\left.-\hat{\psi}_\mu(z)\partial_z\hat{\psi}_m(z)\right]_{z_1}^{z_2}, \tag{15}$$

and

$$\hat{b}_{\mu m}(x,y) = \int_0^{z_1}\hat{\psi}_m(z)\hat{\psi}_\mu(z)\mathrm{d}\,z$$

$$+\int_{z_2}^{\delta}\hat{\psi}_m(z)\hat{\psi}_\mu(z)\mathrm{d}\,z. \tag{16}$$

Two coupling mechanisms can be identified, namely the "bulk" or "global" modal coupling and the "boundary" or "local" modal coupling [10]. The term $\hat{b}_{\mu m}$, is related to the coupling of modes throughout the section of the guide (arising from the non orthogonality of the modes in the perturbed lateral dimensions of the guide due to the depth of the surface perturbation), while the operator $\hat{\gamma}_{\mu m}$, is related to the coupling through the slope and the depth of the surface perturbation itself. The behaviour of the acoustic pressure field is determined by these two mechanisms when propagating along the axis of the waveguide, the continuously distributed modes coupling along the distributed slight geometrical perturbation being accounted for in using method relying on integral formulation.

3 Analytical results

3.1 Approximate integral solution

The coefficients $\hat{A}_m$ are determined using methods relying on integral formulation and modal analysis, and using the appropriate Green function G_m [8].

When the waveguide is infinite and bounded by surfaces with one dimensional corrugations (two-dimensional problem $z\in\left[z_1,z_2\right]$ and $x\in\left[0,+\infty\right[$) the appropriate one-dimensional Green's function corresponding to a point source located at a point in the waveguide is given by

$$G_m(x;x')=\frac{e^{-ik_{x_m}|x-x'|}}{2ik_{x_m}}. \tag{17}$$

The solution is obtained from an iterative method to express the amplitude $\hat{A}_m$ of each mode, the lower order (Born approximation), $\hat{A}_m^{(0)}$ being given by

$$\hat{A}_m^{(0)}(x) = \hat{Q}_m G_m(x;x'), \tag{18}$$

where $\hat{Q}_m$ is the strength of a monochromatic source which is assumed to be flush-mounted at $x=0$, related to the m-th mode; in fact, it represents the energy transfer between the external source and the eigenmode m. The first-order perturbation expansion $\hat{A}_m^{(1)}$ is then given by

$$\hat{A}_m^{(1)}(x) = -\sum_\mu\int_{-\infty}^{+\infty}G_m(x;x')\hat{\gamma}_{\mu m}(x')\hat{A}_\mu^{(0)}(x')\mathrm{d}\,x'$$

$$+\sum_\mu\hat{Q}_\mu\int_{-\infty}^{+\infty}G_m(x;x')\hat{b}_{\mu m}(x')\mathrm{d}\,x'$$

$$-\sum_\mu\left(\hat{k}_{x_m}^2-\hat{k}_{x_\mu}^2\right)\int_{-\infty}^{+\infty}G_m(x;x')\hat{b}_{\mu m}(x')\hat{A}_\mu^{(0)}(x')\mathrm{d}\,x'. \tag{19}$$

3.2 Phonon relation

As far as spatial periods Λ, for a periodic roughness, are concerned, relationships between both the acoustic wavelengths along the x-axis ($\lambda_{x_m}=2\pi/k_{x_m}$ for mode m generated by the source and $\lambda_{x_\mu}=2\pi/k_{x_\mu}$ for modes μ created by the scattering on the corrugation) and the length of the spatial period Λ appear, involving a phase matching which emphasizes the interference processes (phonon relations [11]), namely (for example)

$$\hat{k}_{x_m}+\hat{k}_{x_\mu}\pm 2\pi/\Lambda=0, \tag{20}$$

i.e., using Eq. (12),

$$\frac{\hat{k}_{x_\mu}d}{2\pi}=\pm\frac{d}{\Lambda}-\sqrt{\left(\frac{fd}{c_0}\right)^2-\left(\frac{m}{2}\right)^2}, \tag{21}$$

and showing therefore a strong coupling between the primary wave (wavenumber k_{x_m}) and the counter-propagating secondary wave (wavenumber $-k_{x_\mu}$) for this example.

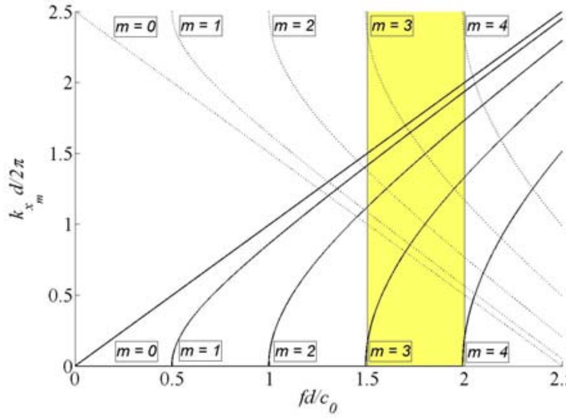


Fig.2 Dispersion curves (solid lines) of the guide with smooth interfaces and curves (dashed lines) corresponding to the phonon relation (21).

4 Numerical results

4.1 Modes coupling

The roughness profile is a sawtooth profile. The corrugation starts at the abscissa $x_0 = 0$, and length ℓ of the corrugation is such as $k_0\ell = 823$ which corresponds to $N = 200$ teeth and to $\ell \approx 131\lambda$. The heights of the teeth are such as $h/d = 0.005$.

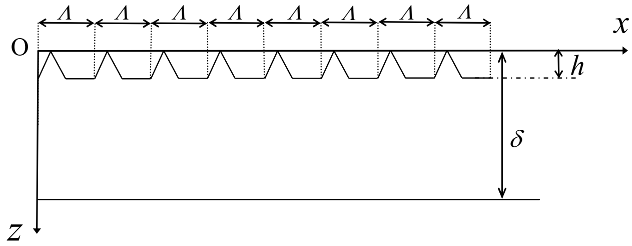


Fig.3 Sawtooth profile.

For $fd/c_0 = 1.30$, the dashed line (Fig .2) which represents the phonon relation (20,21) for $m = 1$ has an intersection with the dispersion curve of the regular-shaped guide corresponding to $\mu = 0$ (labeled $m = 0$ on the figure). Therefore, when the source creates the mode $m = 1$, a strong coupling appears with the mode $\mu = 0$. This is the situation presented in Fig .4 and Fig .5 showing the modulus of the normalised amplitudes calculated at the 3rd order perturbation expansion for a periodic sawtooth profile (Fig .3), without dissipation (blue curve) and with dissipation (red curve). It is assumed that, inside the two-dimensional waveguide bounded by two parallel plates, the only mode created by the source is mode $m = 1$ and that the frequency is such as $fd/c_0 = 1.31$ with $d/\Lambda = 2.5$ so that the upper modes ($\mu > 2$) are evanescent; the fundamental plane mode ($\mu = 0$) is thus the only propagative mode created by coupling due to the corrugation.

Due to the periodicity of the corrugation, periods appear in Fig .4 which are given by the following relationships (from the shorter period to the longer one)

$$2\pi k_0 \Big/ \frac{1}{2}\left(\hat{k}_{x_{m=1}} + \frac{2\pi}{\Lambda} + \hat{k}_{x_{m=0}}\right),$$

$$2\pi k_0 \Big/ \frac{1}{2}\left(\hat{k}_{x_{\mu=0}} - \hat{k}_{x_{m=1}}\right),$$

$$2\pi k_0 \Big/ \frac{1}{2}\left(\hat{k}_{x_{m=1}} - \frac{2\pi}{\Lambda} + \hat{k}_{x_{\mu=0}}\right).$$

The modulus of the normalized amplitude of the mode $m = 1$ decreases when the abscissa increase; it provides a small part of his own energy to the mode $\mu = 0$ by coupling.

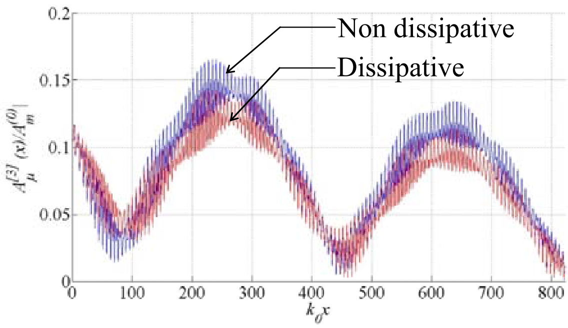


Fig.4 Modulus of the normalized amplitude $\hat{A}^{[3]}_{\mu=0}(x)\big/\hat{A}^{(0)}_{m}$ of the mode $\mu = 0$ for a sawtooth profile, as a function of the abcissa of the corrugation, without dissipation (upper blue curve) and with dissipation (lower red curve).

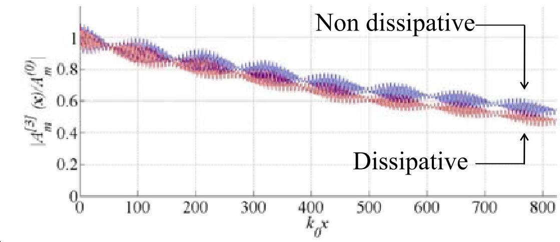


Fig.5 Modulus of the normalized amplitude $\hat{A}^{[3]}_{m=1}(x)\big/\hat{A}^{(0)}_{m}$ of the mode $m = 1$ for a sawtooth profile, as a function of the abcissa of the corrugation, without dissipation (upper blue curve) and with dissipation (lower red curve).

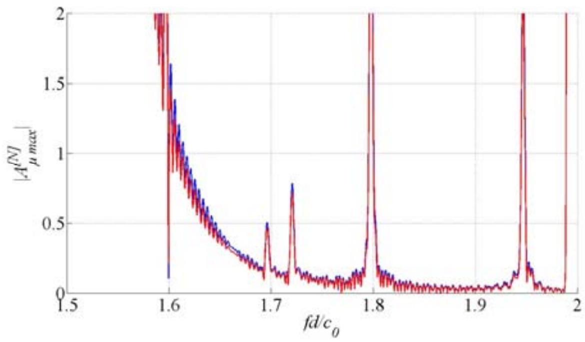
4.2 Phonon relation

For the same waveguide, the modulus of the normalized amplitude of the mode $\mu = 3$ at the abscissa $x = \ell$, calculated to the 5th order of perturbation, is shown as a function of the frequency in Fig .6. The profile of the corrugation is sinusoidal and contains $N = 200$ periods. The ratio of the heights of the sine shape and the thickness of the plate is such as $h/d = 0.005$.

The amplitudes are calculated for a frequency included in the interval between (yellow band in Fig .2) $fd/c_0 = 1.50$ and $fd/c_0 = 2$, these limits of the interval corresponding to frequencies close to the cut off frequencies respectively of the modes $\mu = 3$ and $\mu = 4$. Phonon relationships between the mode $\mu = 3$ and the modes $\mu = 0,1,3$, and $m = 2$ are in this interval. Another phonon relation concerning the mode $m = 2$ with itself is also in this interval.

The modulus of the normalized amplitude of the mode $\mu = 3$ increases close to the frequencies $fd/c_0 = 1.70$, $fd/c_0 = 1.72$, $fd/c_0 = 1.80$ and $fd/c_0 = 1.85$, which, according to Fig .2, are repectively the frequencies of the phonon relations between $\mu = 3$ and $\mu = 0$, $\mu = 3$ and $\mu = 1$, $\mu = 3$ and $m = 2$ and $\mu = 3$ with itself.

The phonon relation of the mode $m = 2$ with itself is located at the frequency $fd/c_0 = 1.60$ which is close to the cut off frequency of the mode $\mu = 3$.



.Fig.6 Modulus of the normalized amplitude $\hat{A}^{[5]}_{\mu=3}(x) / \hat{A}^{(0)}_{m}$ of the mode $\mu = 3$ for a sinusoidal profile, as a function of the frequency, without dissipation (upper blue curve) and with dissipation (lower red curve).

5 Conclusion

A model describing the inter-modal couplings in three dimensional waveguides with non homogeneously-shaped walls (which includes the slopes and the depth of the corrugation), available in the literature when the dissipation phenomena are not accounted for (Neumann boundary conditions), is extended in order to take into account the losses inside the thermo-viscous boundary layers (mixed boundary conditions).

The obtained results confirm the strong coupling when phonon relations occur, and emphasize the effect of the dissipative character of the fluid. In particular, the losses strengthen the decreasing of the amplitude of the mode created by the acoustic source.

Acknowledgments

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