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General absorbing boundary for acoustic and elastic waves

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In the finite element computation of unbounded acoustic problems, the domains must be of finite size and artificial absorbing conditions have to be introduced in order to avoid reflections at the truncated boundary. This communication proposes a new method in the frequency domain to generate efficient absorbing boundary conditions without the need to consider high order derivatives on the boundary. Moreover the approach is quite general and can consider media with mean flow. It needs the knowledge of the dynamic stiffness matrix of only one element which can be obtained from any finite element software. The final result is an impedance condition linking the forces and displacements on the boundary which is determined numerically from a wave analysis of one element of the medium. Using this impedance condition a finite element problem can be solved by standard methods. This is then applied to study an example for a 2D acoustic problem with mean flow.

1 Introduction

Wave problems in unbounded media can occur in many applications in acoustics and mechanics. For methods solving the problem on a bounded domain like the finite element method, it introduces the difficulty of an artificial boundary to get a bounded domain. This boundary must be such that the energy crosses it without reflection and special conditions must be specified at the artificial boundary to reproduce this phenomena. Generally these can be classified into local or global boundary conditions. The first global method which has been used for solving such problems was the boundary element method. This method is well adapted for infinite domains and is described in classical textbooks like [1]. In the other approaches, the computational domain is truncated at some distance and boundary conditions are imposed at this artificial boundary. An example of global boundary condition is the Dirichlet to Neumman (DtN) mapping proposed by [2]. It consists in dividing the domain into a finite part containing the sources and an infinite domain of simple shape. The DtN mapping is non local and every node on the boundary is connected to all other nodes. Other methods are local and the condition at a node involves only neighbouring nodes. A first possibility of such approaches is the use of infinite elements as proposed in [3]. It consists in developing special elements with a behaviour at infinity reflecting that of analytical solutions obtained for the same type of problems. In the perfectly matched layer proposed by [4], an exterior layer of finite thickness is introduced around the bounded domain. The absorption in this domain is increasing as we move towards the exterior such that outgoing waves are absorbed before reaching the exterior boundary.

In the proposed method, the boundary condition is described in term of waves but the final expression involves only the variable and its derivative. The approach is based on the waveguide theory for periodic media described in [5, 6, 7]. Only information related to one period, obtained from any standard FE software (the discrete dynamic stiffness matrices and nodal coordinates) are required to formulate the method. The advantage of the method proposed here is that it can be applied to media with complex behaviours. In section 2, the methodology for determining absorbing boundary conditions for finite size periodic media is described. In section 3, an application is described to show the results of the method for the finite element computation of acoustic propagation with mean flow.

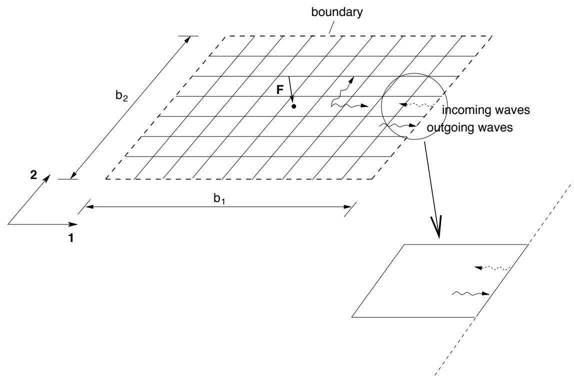


Figure 1: Periodic medium.

2 Absorbing boundary conditions

We suppose that we want to solve a mechanical problem on an infinite domain exterior to a bounded domain Ω_{int}. The infinite domain is approximated by the finite domain Ω which is exterior to Ω_{int} and is limited by an exterior boundary Γ_{ext}. Near this exterior boundary the solution can be seen as composed of incident waves denoted A_+ and reflected waves A_-. For a perfectly absorbing boundary, one should have $A_- = 0$. The approach proposed in this paper consists in studying this problem by first considering the case of periodic media. For this case, positive and negative waves and their amplitudes A_+ and A_- can be computed. Then an exact boundary condition can be formulated for a half-plane boundary. It is further shown how this condition can be approximated by a local condition on the boundary.

2.1 Solution in a periodic medium

Consider an infinite two dimensional periodic medium, as shown in figure 1. The elementary period is limited by the domain $(x_1, x_2) \in [0, b_1] \times [0, b_2]$. A function $U(x_1, x_2)$ defined on the two-dimensional periodic medium can be decomposed as an integral of pseudo periodic functions

$$U(x_1, x_2) = \int_{-\frac{\pi}{b_2}}^{\frac{\pi}{b_2}} e^{ikx_2} \hat{U}(x_1, k, x_2) dk \qquad (1)$$

where $\hat{U}(x_1, k, x_2)$ is a periodic function in x_2 with period b_2. From the inverse Fourier transform one also has

$$\hat{U}(x_1,k,x_2) = \frac{b_2}{2\pi}\sum_{m_2=-\infty}^{+\infty} e^{-ik(x_2+m_2b_2)}U(x_1,x_2+m_2b_2) \tag{2}$$

From Eq.(1), one sees that the behaviour in x_2 of the general solution can be obtained from functions such that $e^{ikx_2}\hat{U}(x_1,k,x_2)$ with $\hat{U}(x_1,k,x_2)$ periodic in x_2. Along direction 1, we use a decomposition in Bloch waves as it is usual in periodic media. Finally, the general solution can be obtained from functions $u(x_1,k,x_2) = e^{ikx_2}\hat{U}(x_1,k,x_2)$ such that:

$$\begin{aligned} u(x_1,k,x_2+m_2b_2) &= e^{ikm_2b_2}u(x_1,k,x_2) \\ u(x_1+m_1b_1,k,x_2) &= e^{im_1\mu}u(x_1,k,x_2) \end{aligned} \tag{3}$$

where m_1 and m_2 are integers, $k \in \mathrm{R} \cap \left[-\frac{\pi}{b_2},\frac{\pi}{b_2}\right]$ and $\mu \in \mathrm{C}$.

The discrete dynamic equation of a cell (an elementary period) obtained from a FE model at a frequency ω and for the time dependence $e^{-i\omega t}$ is given by:

$$(\mathbf{K} - i\omega\mathbf{C} - \omega^2\mathbf{M})\mathbf{q} = \mathbf{f} \tag{4}$$

where $\mathbf{K}$, $\mathbf{M}$ and $\mathbf{C}$ are the stiffness, mass and damping matrices, respectively, $\mathbf{f}$ is the loading vector and $\mathbf{q}$ the vector of the degrees of freedom (dofs). Introducing the dynamic stiffness matrix $\widetilde{\mathbf{D}} = \mathbf{K} - i\omega\mathbf{C} - \omega^2\mathbf{M}$, decomposing the dofs into boundary (B) and interior (I) dofs, and assuming that there are no external forces on the interior nodes, yields

$$\begin{bmatrix} \widetilde{\mathbf{D}}_{BB} & \widetilde{\mathbf{D}}_{BI} \\ \widetilde{\mathbf{D}}_{IB} & \widetilde{\mathbf{D}}_{II} \end{bmatrix}\begin{bmatrix} \mathbf{q}_B \\ \mathbf{q}_I \end{bmatrix} = \begin{bmatrix} \mathbf{f}_B \\ \mathbf{0} \end{bmatrix} \tag{5}$$

The interior dofs can be eliminated which yields

$$\mathbf{f}_B = \left(\widetilde{\mathbf{D}}_{BB} - \widetilde{\mathbf{D}}_{BI}\widetilde{\mathbf{D}}_{II}^{-1}\widetilde{\mathbf{D}}_{IB}\right)\mathbf{q}_B \tag{6}$$

which can be written as

$$\mathbf{f} = \mathbf{D}\mathbf{q} \tag{7}$$

It should be noted that only boundary dofs are considered in the following sections.

The periodic cell is assumed to be meshed with an equal number of nodes on their opposite sides. The boundary dofs are decomposed into left (L), right (R), bottom (B), top (T) dofs and associated corners (LB), (RB), (LT) and (RT) as shown in figure 2. The longitudinal dofs vector is defined as

$$\mathbf{q}_l = {}^t\left[{}^t\mathbf{q}_L \quad {}^t\mathbf{q}_R \quad {}^t\mathbf{q}_{LB} \quad {}^t\mathbf{q}_{RB} \quad {}^t\mathbf{q}_{RT} \quad {}^t\mathbf{q}_{LT}\right] \tag{8}$$

where ^{t} means the transpose. Eq.(7) is rewritten as

$$\begin{bmatrix} \mathbf{D}_{ll} & \mathbf{D}_{lB} & \mathbf{D}_{lT} \\ \mathbf{D}_{Bl} & \mathbf{D}_{BB} & \mathbf{D}_{BT} \\ \mathbf{D}_{Tl} & \mathbf{D}_{TB} & \mathbf{D}_{TT} \end{bmatrix}\begin{bmatrix} \mathbf{q}_l \\ \mathbf{q}_B \\ \mathbf{q}_T \end{bmatrix} = \begin{bmatrix} \mathbf{f}_l \\ \mathbf{f}_B \\ \mathbf{f}_T \end{bmatrix} \tag{9}$$

Using the pseudo periodic condition Eq.(3) and the effort equilibrium at the bottom side of the cell, relations between the transverse dofs are given by

$$\begin{aligned} \mathbf{q}_T &= e^{ikb_2}\mathbf{q}_B \\ \mathbf{f}_B + e^{-ikb_2}\mathbf{f}_T &= 0 \end{aligned} \tag{10}$$

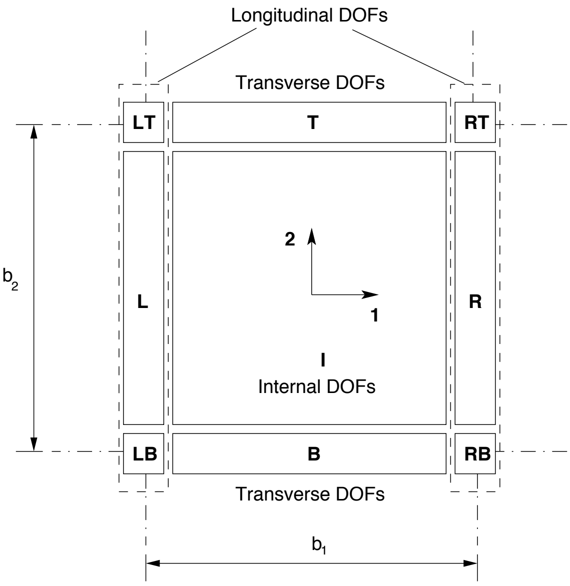


Figure 2: A cell in the periodic medium.

Multiplying the third row of Eq.(9) with e^{-ikb_2}, taking the sum of the second and third rows of Eq.(9), using Eq.(10), lead to

$$\begin{aligned} \mathbf{f}_l = {} & [\mathbf{D}_{ll} - (\mathbf{D}_{lB} + e^{ikb_2}\mathbf{D}_{lT})(\mathbf{D}_{BB} + \mathbf{D}_{TT} \\ & + e^{-ikb_2}\mathbf{D}_{TB} + e^{ikb_2}\mathbf{D}_{BT})^{-1} \\ & \times(\mathbf{D}_{Bl} + e^{-ikb_2}\mathbf{D}_{Tl})]\mathbf{q}_l = \mathbf{D}_l\mathbf{q}_l \end{aligned} \tag{11}$$

Using the pseudo periodic conditions Eq.(3) also lead to the following relations between longitudinal dofs

$$\begin{aligned} \mathbf{q}_R &= e^{i\mu}\mathbf{q}_L \\ \mathbf{q}_{RB} &= e^{i\mu}\mathbf{q}_{LB} \\ \mathbf{q}_{RT} &= e^{i(\mu+kb_2)}\mathbf{q}_{LB} \\ \mathbf{q}_{LT} &= e^{ikb_2}\mathbf{q}_{LB} \end{aligned} \tag{12}$$

From the pseudo periodic conditions Eq.(12), it can be seen that all components of the vector $\mathbf{q}_l$ depend on the set of dofs defined by $\mathbf{q}_r = {}^t\left[{}^t\mathbf{q}_L \quad {}^t\mathbf{q}_{LB}\right]$. This can be expressed as

$$\mathbf{q}_l = \left(\mathbf{W}_0 + e^{i\mu}\mathbf{W}_1\right)\mathbf{q}_r \tag{13}$$

where the matrices $\mathbf{W}_0$ and $\mathbf{W}_1$ depend on the wavenumber k and are given by

$$\mathbf{W}_0 = \begin{bmatrix} \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \\ \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \\ \mathbf{O} & e^{ikb_2}\mathbf{I} \end{bmatrix} \qquad \mathbf{W}_1 = \begin{bmatrix} \mathbf{O} & \mathbf{O} \\ \mathbf{I} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \\ \mathbf{O} & \mathbf{I} \\ \mathbf{O} & e^{ikb_2}\mathbf{I} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \tag{14}$$

The equilibrium conditions between adjacent cells are given by

$$\begin{aligned} e^{i\mu}\mathbf{f}_L + \mathbf{f}_R &= 0 \\ e^{i\mu}\mathbf{f}_{LB} + \mathbf{f}_{RB} + e^{i(\mu-kb_2)}\mathbf{f}_{LT} + e^{-ikb_2}\mathbf{f}_{RT} &= 0 \end{aligned} \tag{15}$$

that can be written as

$$\left(e^{i\mu}\mathbf{W}_0^* + \mathbf{W}_1^*\right)\mathbf{f}_l = 0 \tag{16}$$

where $(.)^*$ denotes the operator of complex conjugate and transpose.

Combining Eq.(11), Eq.(13) and Eq.(16), lead to

$$\left(\mathbf{A}_0 + e^{i\mu}(\mathbf{A}_1 + \mathbf{A}_2) + e^{2i\mu}\mathbf{A}_3\right)\mathbf{q}_r = 0 \quad (17)$$

where

$$\begin{array}{rcl} \mathbf{A}_0 & = & \mathbf{W}_1^*\mathbf{D}_l\mathbf{W}_0 \\ \mathbf{A}_1 & = & \mathbf{W}_0^*\mathbf{D}_l\mathbf{W}_0 \\ \mathbf{A}_2 & = & \mathbf{W}_1^*\mathbf{D}_l\mathbf{W}_1 \\ \mathbf{A}_3 & = & \mathbf{W}_0^*\mathbf{D}_l\mathbf{W}_1 \end{array} \quad (18)$$

The eigenvalue $e^{i\mu}$ and the eigenvector $\mathbf{q}_r$ are thus solutions of a quadratic eigenvalue problem. It can be easily shown that if $e^{i\mu_j}$ is an eigenvalue for the wavenumber k, $e^{-i\mu_j}$ is also an eigenvalue for the wavenumber $-k$. These represent a pair of positive and negative-going waves, respectively. Thus, $2n$ eigensolutions of Eq. (17) can be split into two sets of n^+ and n^- eigensolutions with $2n = n^+ + n^-$, which are denoted by $\left(e^{i\mu_j^+}, \mathbf{q}_j^+\right)$ and $\left(e^{i\mu_j^-}, \mathbf{q}_j^-\right)$ respectively, with the first set such that $\left|e^{i\mu_j^+}\right| \leq 1$. In the case $\left|e^{i\mu_j^+}\right| = 1$, the first set of positive-going waves must contain waves propagating in the positive direction such that $\text{Re}\left\{i\omega\mathbf{q}_j^H\mathbf{f}_j^r\right\} > 0$ where $\mathbf{f}_j^r$ is the reduced set of boundary force dofs of left cells on right cells and is given by

$$\begin{array}{rcl} \mathbf{f}_j^r & = & \left[\begin{array}{c} \mathbf{f}_L \\ \mathbf{f}_{LB} + e^{-ikb_2}\mathbf{f}_{LT} \end{array}\right] \\ & = & \mathbf{W}_0^*\mathbf{D}_l\left(\mathbf{W}_0 + e^{i\mu_j}\mathbf{W}_1\right)\mathbf{q}_j \end{array} \quad (19)$$

In the second set of negative-going waves, the eigenvalue $e^{i\mu_j^-}$ is associated with waves such that $\text{Re}\left\{i\omega\mathbf{q}_j^H\mathbf{f}_j^r\right\} < 0$. With the eigenvector $\mathbf{q}_j$ and the force component of Eq.(19), we introduce the state vector

$$\mathbf{x}_j(k) = \left[\begin{array}{c} \mathbf{q}_j(k) \\ \mathbf{f}_j^r(k) \end{array}\right] = \left[\begin{array}{c} \mathbf{q}_j(k) \\ (\mathbf{A}_1(k) + e^{i\mu_j(k)}\mathbf{A}_3(k))\mathbf{q}_j(k) \end{array}\right] \quad (20)$$

In this relation $\mathbf{q}_j(k)$ is the eigenvector associated to $e^{i\mu_j(k)}$. One can also introduce

$$\mathbf{y}_j(-k) = \left[\begin{array}{cc} {}^t\mathbf{p}_j(-k)(\mathbf{A}_2(k) + e^{i\mu_j(k)}\mathbf{A}_3(k)) & {}^t\mathbf{p}_j(-k) \end{array}\right] \quad (21)$$

In this relation $\mathbf{p}_j(-k)$ is the eigenvector associated to $e^{-i\mu_j(k)}$ since we have seen that $e^{-i\mu_j(k)}$ is also an eigenvalue of Eq.(17) for the wavenumber $-k$. It is possible to compute the product $\mathbf{y}_i(-k).\mathbf{x}_j(k)$ to show that

$$\mathbf{y}_i(-k).\mathbf{x}_j(k) = d_i\delta_{ij} \quad (22)$$

d_i is a factor depending on the eigenvector i. This gives orthogonality relations on the statevectors associated to the eigenvalues.

2.2 Absorbing boundary conditions

Near the exterior boundary, the solution is described by Eq.(1). Introducing the state vector $\mathbf{x} = {}^t({}^t\mathbf{q}, {}^t\mathbf{f})$ and decomposing this solution into the different waves, we get

$$\begin{array}{rcl} \mathbf{x}(x_1, x_2) & = & \displaystyle\int_{-\frac{\pi}{b_2}}^{\frac{\pi}{b_2}} \hat{\mathbf{x}}(x_1, k, x_2)e^{ikx_2}dk \\ & = & \displaystyle\int_{-\frac{\pi}{b_2}}^{\frac{\pi}{b_2}} \sum_{j=1}^{j=2n} a_j(x_1, k)\mathbf{x}_j(k)e^{ikx_2}dk \end{array} \quad (23)$$

The condition of outgoing waves means that there is no incoming wave, so the amplitudes $a_j(x_1, k)$ associated with incoming waves must equal zero. This condition is obtained by

$$\mathbf{y}_l^-(-k). \sum_{j=1}^{j=2n} a_j(x_1, k)\mathbf{x}_j(k) = 0 \quad for\ 1 \leq l \leq n^- \quad (24)$$

In this relation $\mathbf{y}_l^-(-k)$ are the vectors associated to the negative going waves, given by Eq.(21). Using Eq.(22), one gets $a_j^-(x_1, k) = 0$ with $1 \leq j \leq n^-$ for the amplitudes of the negative going waves. Introducing the matrix $\mathbf{Y}$ with lines given by $\mathbf{y}_l$ leads to

$$\mathbf{Y}(-k).\hat{\mathbf{x}}(x_1, k, x_2) = 0 \quad (25)$$

Decomposing now $\hat{\mathbf{x}}$ into its displacement and force components, doing the same thing for $\mathbf{Y}(-k)$ with $\mathbf{Y}(-k) = [\mathbf{Q}(-k) \quad \mathbf{F}(-k)]$ leads to

$$\mathbf{Q}(-k).\hat{\mathbf{q}}(x_1, k, x_2) + \mathbf{F}(-k).\hat{\mathbf{f}}(x_1, k, x_2) = 0 \quad (26)$$

then from Eq.(23)

$$\mathbf{f}(x_1, x_2) = -\int_{-\frac{\pi}{b_2}}^{\frac{\pi}{b_2}} \mathbf{F}^{-1}(-k)\mathbf{Q}(-k)\hat{\mathbf{q}}(x_1, k, x_2)e^{ikx_2}dk \quad (27)$$

From the inverse Eq.(2), one also has

$$\hat{\mathbf{q}}(x_1, k, x_2) = \frac{b_2}{2\pi} \sum_{m_2=-\infty}^{+\infty} e^{-ik(x_2+m_2b_2)}\mathbf{q}(x_1, x_2 + m_2b_2) \quad (28)$$

which leads to

$$\begin{array}{rcl} \mathbf{f}(x_1, x_2) & = & \displaystyle -\frac{b_2}{2\pi}\int_{-\frac{\pi}{b_2}}^{\frac{\pi}{b_2}} \mathbf{F}^{-1}(-k)\mathbf{Q}(-k) \\ & & \displaystyle \sum_{m_2=-\infty}^{+\infty} e^{-ik(x_2+m_2b_2)}\mathbf{q}(x_1, x_2 + m_2b_2)dk \end{array} \quad (29)$$

Introducing the function

$$\mathbf{G}(x_2) = -\frac{b_2}{2\pi}\int_{-\frac{\pi}{b_2}}^{\frac{\pi}{b_2}} \mathbf{F}^{-1}(-k)\mathbf{Q}(-k)e^{-ikx_2}dk \quad (30)$$

The final relation is

$$\mathbf{f}(x_1, x_2) = \sum_{m_2=-\infty}^{+\infty} \mathbf{G}(x_2 + m_2b_2)\mathbf{q}(x_1, x_2 + m_2b_2) \quad (31)$$

This is the impedance relation on the boundary obtained with the assumption that there is no negative going wave. This relation is the absorbing boundary condition we were looking for. It can be computed from the wave vectors and the force components associated with them. Up to now everything has been written for periodic media but it is clear that homogeneous media are also periodic media and so all that has been said applies also to homogeneous media.

Eq.(31) involves an infinite number of terms on the boundary. For practical purposes we will use the approximate relations at various orders

$$\begin{array}{l} \mathbf{f}(x_1, x_2) \approx \mathbf{G}_0\mathbf{q}(x_1, x_2) \\ +\dfrac{\mathbf{G}_1}{2b_2}(\mathbf{q}(x_1, x_2 + b_2) - \mathbf{q}(x_1, x_2 - b_2)) \\ +\dfrac{\mathbf{G}_2}{2b_2^2}(\mathbf{q}(x_1, x_2 + b_2) + \mathbf{q}(x_1, x_2 - b_2) - 2\mathbf{q}(x_1, x_2)) \\ + \ldots \end{array} \quad (32)$$

with

$$\begin{aligned}
\mathbf{G}_0 &= \sum_{m_2=-\infty}^{+\infty} \mathbf{G}(x_2 + m_2 b_2) = -(\mathbf{F}^{-1}\mathbf{Q})(0) \\
\mathbf{G}_1 &= \sum_{m_2=-\infty}^{+\infty} m_2 b_2 \mathbf{G}(x_2 + m_2 b_2) = i(\mathbf{F}^{-1}\mathbf{Q})'(0) \\
\mathbf{G}_2 &= \sum_{m_2=-\infty}^{+\infty} (m_2 b_2)^2 \mathbf{G}(x_2 + m_2 b_2) = (\mathbf{F}^{-1}\mathbf{Q})''(0)
\end{aligned} \tag{33}$$

which involves a finite number of nodes around the point where the relation has to be written.

3 Example

3.1 Validation example

Consider as an example an acoustic element of size $0.01m \times 0.01m$ or $0.05m \times 0.05m$, a sound velocity $c = 340m/s$ and a plane wave excitation with an incidence angle $\theta = 10^o$. The pressure is suppose given and we compare its normal derivative given respectively by the analytical and the approximate Eq.(32). The relative errors at point $(0, 0)$ between these two quantities are plotted in figure 3. It can be observed that the second order relations are much better than the first ones as expected. The comparison of the two sizes for the acoustic element shows that the size $0.05m \times 0.05m$ can reduce the accuracy of the solution for high frequencies. In these cases it is better to use the small size for the element.

The error is also plotted versus the angle of incidence of the plane wave in figure 4. The solution is accurate (error less than 1%) for angles up to 10^o for a zero order condition and up to 30^o for a second order condition. Finally the error is plotted versus the distance along the y axis for a pressure created by a point source at point $(-1, 0)$ and at the frequency $1000Hz$. The reduction in accuracy can be observed as we move along the y axis leading to greater incidence angles. All these points confirm the accuracy of the method proposed here.

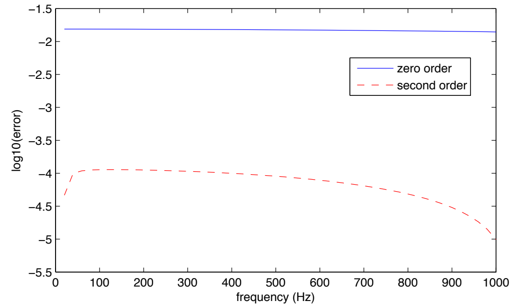
3.2 Case with mean flow

The precedent method is applied here to the computation of an acoustic propagation problem with mean flow. Sound propagation in a uniform moving medium can be described by the following set of linear equations

$$\begin{aligned}
\frac{\partial p}{\partial t} + \mathbf{v}_0.\nabla p + \rho_0 c^2 \nabla.\mathbf{v} &= q \\
\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}_0.\nabla \mathbf{v} + \frac{1}{\rho_0}\nabla p &= \mathbf{f}
\end{aligned} \tag{34}$$

where $\mathbf{v}$ is the particle velocity vector, p is the acoustic pressure, $\mathbf{v}_0$ is the velocity vector of the fluid flow, c is the sound velocity, ρ_0 the density and q, $\mathbf{f}$ are source terms. The wave equation on p is given, for a point source excitation, by

$$\Delta p - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + \mathbf{v}_0.\nabla\right)^2 p = -\delta(x)\delta(y) \tag{35}$$



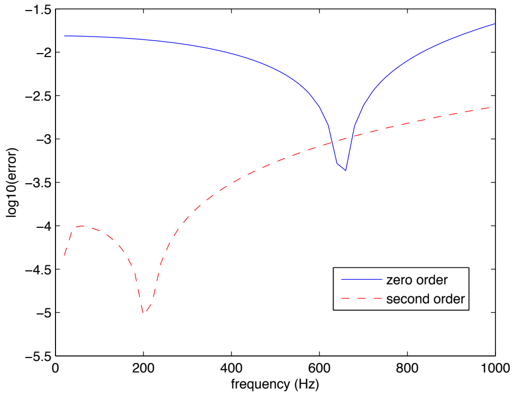


Figure 3: Error versus the frequency for a plane wave at 10^o for an element size $0.01m \times 0.01m$ (upper) and an element size $0.05m \times 0.05m$ (lower).

The analytical solution of this equation, with $\mathbf{v}_0 = (v_x, v_y)$ at an angle α with the point (x, y), is given by

$$p(x,y) = \frac{i}{4(1-M^2)^{1/2}} H_0(\xi) e^{-\frac{ikMr\cos\alpha}{1-M^2}} \tag{36}$$

with

$$\xi = \frac{kr\sqrt{1 - M^2 \sin^2\alpha}}{1 - M^2}, \quad M = \frac{|\mathbf{v}_0|}{c} \tag{37}$$

For the finite element model, the elementary stiffness and mass matrices of a rectangular element of size $b_1 \times b_2$ are given by

$$\begin{aligned}
\mathbf{K} &= \frac{b_2}{6b_1}\left(1 - \frac{v_x^2}{c^2}\right)\begin{bmatrix} 2 & -2 & -1 & 1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ 1 & -1 & -2 & 2 \end{bmatrix} \\
&+ \frac{b_1}{6b_2}\left(1 - \frac{v_y^2}{c^2}\right)\begin{bmatrix} 2 & 1 & -1 & -2 \\ 1 & 2 & -2 & -1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix} \\
&+ \frac{1}{2}\frac{v_x v_y}{c^2}\begin{bmatrix} -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \\
\mathbf{M} &= \frac{b_1 b_2}{36c^2}\begin{bmatrix} 4 & 2 & 1 & 2 \\ 2 & 4 & 2 & 1 \\ 1 & 2 & 4 & 2 \\ 2 & 1 & 2 & 4 \end{bmatrix} \\
\mathbf{C} &= \frac{b_2 v_x}{6c^2}\begin{bmatrix} -2 & 2 & 1 & -1 \\ -2 & 2 & 1 & -1 \\ -1 & 1 & 2 & -2 \\ -1 & 1 & 2 & -2 \end{bmatrix}
\end{aligned}$$

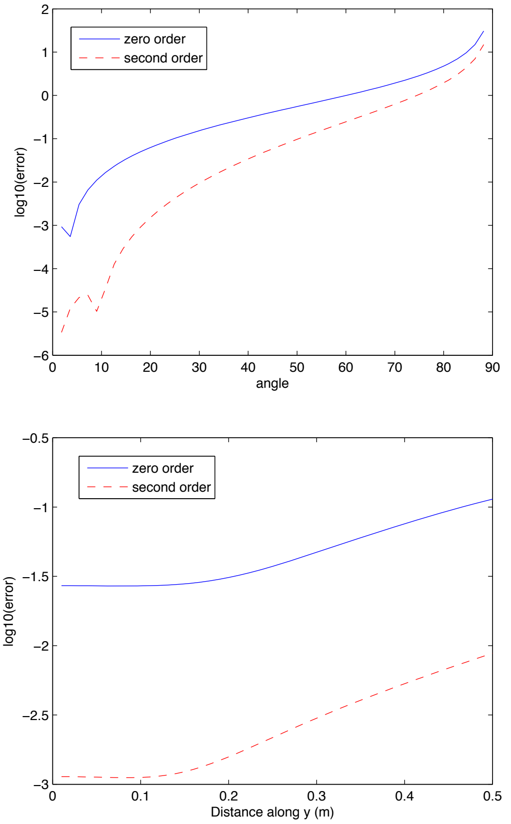


Figure 4: Error versus the angle of incidence (upper) and error versus the distance for a point source (lower).

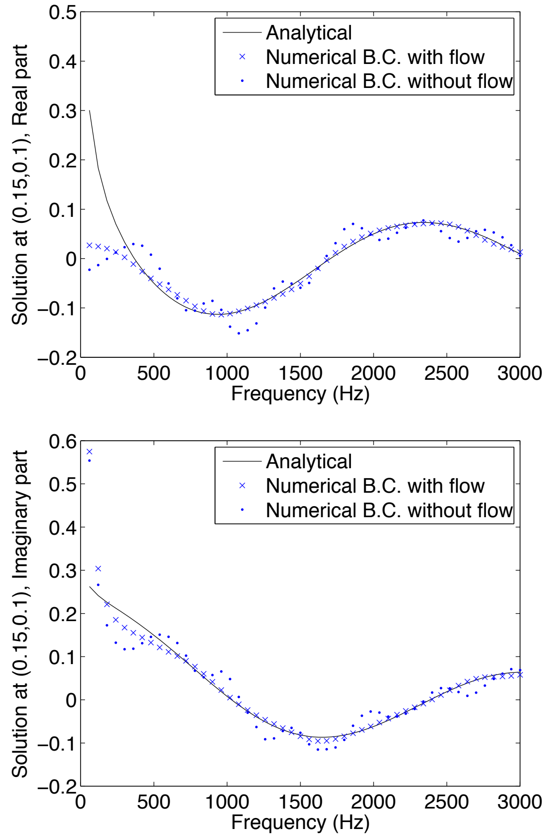


Figure 5: Comparison of analytical and numerical solutions

$$+\frac{b_1 v_y}{6c^2}\begin{bmatrix} -2 & -1 & 1 & 2 \\ -1 & -2 & 2 & 1 \\ -1 & -2 & 2 & 1 \\ -2 & -1 & 1 & 2 \end{bmatrix} \qquad (38)$$

and the dynamic stiffness matrix can then be determined by $\mathbf{D} = \mathbf{K} - i\omega\mathbf{C} - \omega^2\mathbf{M}$.

Consider a square domain of size $0.4m \times 0.4m$ meshed with 80×80 elements of size $0.005m \times 0.005m$. A point source excitation is applied in its centre defined as the origin of the coordinate system. The flow velocity is $\mathbf{v}_0 = (0.6c, 0)$ with $c = 340m/s$. The sound pressure is computed at point $(0.15m, 0.1m)$ and is presented in figure 5. The classical boundary condition $\frac{\partial p}{\partial n} = ikp$ ignoring the velocity of the fluid flow is also presented. It can be observed that the present boundary condition leads to much better results.

4 Conclusion

In this paper, a method to determine absorbing boundary conditions for two-dimensional media has been described. In the example presented, good agreements are observed when compared with analytical solutions. In any case, the proposed method is efficient and general because it requires only the discrete dynamic matrices which can be obtained by any standard FE software.

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