Periodic Assembly of Multi-Coupled Beams: Wave Propagation and Natural Modes

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The present work is concerned with the vibrations of a discrete multi-coupled periodic system. It lies within a larger study on the vibroacoustic behaviour of a heat exchanger. These structures are usually made of a succession of huge number of identical parallel fins (around 600 per meter) connected by tubes conveying the coolant fluid. By now their behaviour can not be calculated using FE model. As a first step, the periodicity principles are applied on a simpler structure, i.e. an assembly of identical beams linked by several damped springs. The basic unit is symmetric and composed of one flexural beam with several springs on each side. Using the Floquet-Bloch's theorem and the works of Denys Mead (receptance matrix), it is possible to completely describe the whole structure behaviour (natural modes, response) only from the vibroacoustic knowledge of the basic unit. This has been done analytically and the results were confirmed by a (very time consuming) FE model calculation. The study of the basic unit can also give valuable information on the physical phenomena governing the transmission from one unit to the next, and then the propagation in the whole structure.

1 Introduction

The present work lies within a larger study, which goal is to build a model of the vibratory behaviour of a heat exchanger, more precisely a finned coil. This element is one of the main parts of air-conditioning systems like liquid chilling package (chillers) or heat pumps. A finned coil is made of large number of identical parallel fins connected to tubes conveying the refrigerant fluid. By now their behaviour can not be calculated using FE model. The regular layout of the fins makes the structure periodic, so this periodicity can be used to simplify the calculation.

The final model of the heat exchanger will consist in a series of parallel plates (fins) connected by several couplings (tubes). In order to get a better knowledge of the phenomena linked with periodicity, the heat exchanger is first reduced to a bidimensional structure, a periodic assembly of parallel beams transversely multi-coupled by springs.

In [1], Mead gave an overview of various methods allowing the analytical formulation of the vibratory behaviour of periodic structures. Among those methods, the ones based on the receptances, on the transfer matrix or using the space-harmonics hold our attention. Mead also contributed to develop the receptance approach [2, 3] for single-coupled periodic structures as well as for multi-coupled periodic structures. The transfer matrix formulation is used to build a model of the assembly of beams.

The first part of this paper described the matrix formulation used to build a model of the vibratory behaviour of the single element. Then the eigenvalues and vectors of the transfer matrix are analysed to highlight how the waves propagate into a periodic assembly of multi-coupled beams. This also permits to observe phenomena linked to the natural modes of the single element.

2 Some theoretical developments about periodic structures

2.1 Matrix formulation

The study of a periodic structure needs first the definition and characterisation of the single element. The duplication of this element will lead to the whole structure. Fig. 1 shows a part of a multi-coupled periodic structure and a diagram of a single element. The subscripts L and R of the displacements q and the forces F respectively refer to the left and right sides of the single element.

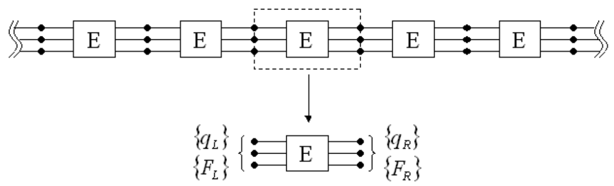


Fig.1 Block diagrams of a periodic structure and a single element.

Modelling the single element consists in writing the relationship between the displacements of the edges and the corresponding forces. This is done with the receptance matrix $[\alpha]$:

$$\begin{Bmatrix} \{q_L\} \\ \{q_R\} \end{Bmatrix} = \begin{bmatrix} [\alpha_{LL}] & [\alpha_{LR}] \\ [\alpha_{RL}] & [\alpha_{RR}] \end{bmatrix} \begin{Bmatrix} \{F_L\} \\ \{F_R\} \end{Bmatrix} \quad (1)$$

Although the analytical formulation of the problem leads naturally to the receptance matrix, the transfer matrix is more convenient to obtain the propagation constants of the single element. This transfer matrix results from the reorganisation of the receptance matrix terms:

$$\begin{Bmatrix} \{q_R\} \\ -\{F_R\} \end{Bmatrix} = \begin{bmatrix} [T_{qq}] & [T_{qF}] \\ [T_{Fq}] & [T_{FF}] \end{bmatrix} \begin{Bmatrix} \{q_L\} \\ \{F_L\} \end{Bmatrix} \quad (2)$$

It leads to a relationship between one side of the single element (displacements and forces) and the other, justifying the concept of transfer.

2.2 Floquet-Bloch's theorem

Wave propagation in periodic structures is based on the use of the Floquet-Bloch's theorem [4]. For a free wave travelling through an infinite periodic structure with the propagation constant μ_j, the theorem gives the link between the displacements and the forces at both sides of the single element:

$$\begin{Bmatrix} \{q_R\} \\ -\{F_R\} \end{Bmatrix} = e^{\mu_j} \begin{Bmatrix} \{q_L\} \\ \{F_L\} \end{Bmatrix} \quad (3)$$

The propagation constants μ_j are obtained from the eigenvalues λ_j of the transfer matrix:

$$\lambda_j = e^{\mu_j} \tag{4}$$

The Floquet-Bloch's theorem thus gives the vibratory behaviour of all the coupling points of an infinite periodic structure from the knowledge of only one of these points.

3 Wave propagation in a periodic assembly of multi-coupled beams

3.1 Structure description

The periodic structure studied in this paper is composed of an infinite number of identical parallel beams linked by three punctual springs of stiffness K. A part of this structure is showed in Fig. 2. For numerical calculations, the beams are made of steel (E = 2.1 10^{11} Pa, ρ = 7800 kg/m³, η = 0.001), with a length of 1 m and a section of 1 × 5 mm². The springs have a stiffness K = 800 N/m and are situated at 17, 38 and 59 cm from the top of the beams. The simply supported boundary conditions allow the use of simple analytical expressions, which will be convenient for the future study of coupled plates.

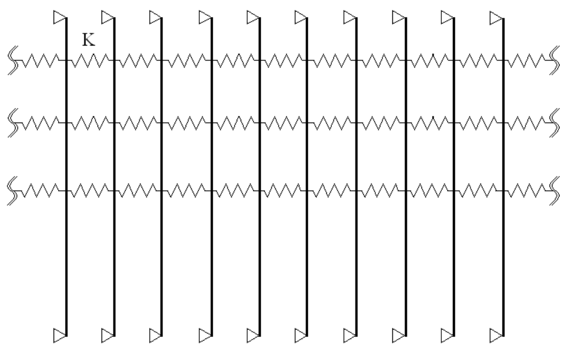


Fig.2 Portion of the infinite periodic assembly.

The whole assembly can be obtained by repeating the single element of Fig. 3. This single element consists of a simply supported beam with the equivalent of half-springs on each side, which makes it symmetric. According to Mead [3], this symmetry leads to a simpler formulation and an easier analysis.

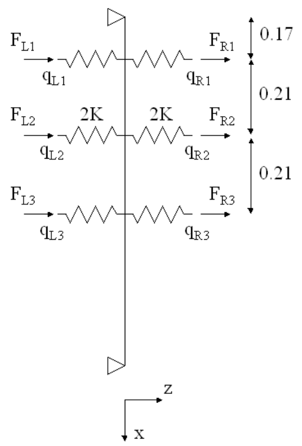


Fig.3 Diagram of the single element.

3.2 Transfer matrix formulation of the single element

Considering the single element of Fig. 3, the calculation of its receptance matrix is based on the flexural beam equation. The elastic couplings are introduced in the way of punctual forces:

$$\tilde{E}I\frac{\partial^4 W(x,t)}{\partial x^4} + \rho S\frac{\partial^2 W(x,t)}{\partial t^2} = \sum_{i=1}^{3}\left[\left(F_{Li} + F_{Ri}\right).\delta\left(x - (id + r - d)\right)\right] \tag{5}$$

with $\tilde{E} = (1 + j\eta)E$.

The beam displacement is then projected onto the natural modes of a simply supported beam:

$$W(x,t) = e^{j\omega t}.\sum_m A_m.\sin\left(\frac{m\pi}{L}x\right) \tag{6}$$

Finally, the single element can be modelled by the following transfer matrix:

$$[T] = \begin{bmatrix} \frac{1}{2K}[G]^{-1} + [I] & -\frac{1}{4K^2}[G]^{-1} - \frac{1}{K}[I] \\ -[G]^{-1} & \frac{1}{2K}[G]^{-1} + [I] \end{bmatrix} \tag{7}$$

where $G_{ij} = \sum_p \frac{1}{H_p}.\sin\left(\frac{p\pi}{L}(id + r - d)\right)\sin\left(\frac{p\pi}{L}(jd + r - d)\right)$

with *i, j = 1 to 3*, $[I]$ being a 3×3 identity matrix.

3.3 Propagation constants

The eigenvalues of the transfer matrix Eq.(7) lead to the propagation constants plotted on Fig. 4. Each propagation constant is associated with a wave travelling in the structure. These propagation constants being complex, both real and imaginary parts have to be analysed.

It can be noticed that the propagation constants appear by pairs, one corresponding to the propagation in the positive direction (dashed line) and the other one corresponding to the propagation in the negative direction (continuous line). Fig. 4 also shows that there are as many pairs of propagation constants as couplings between single elements (three for the case under study).

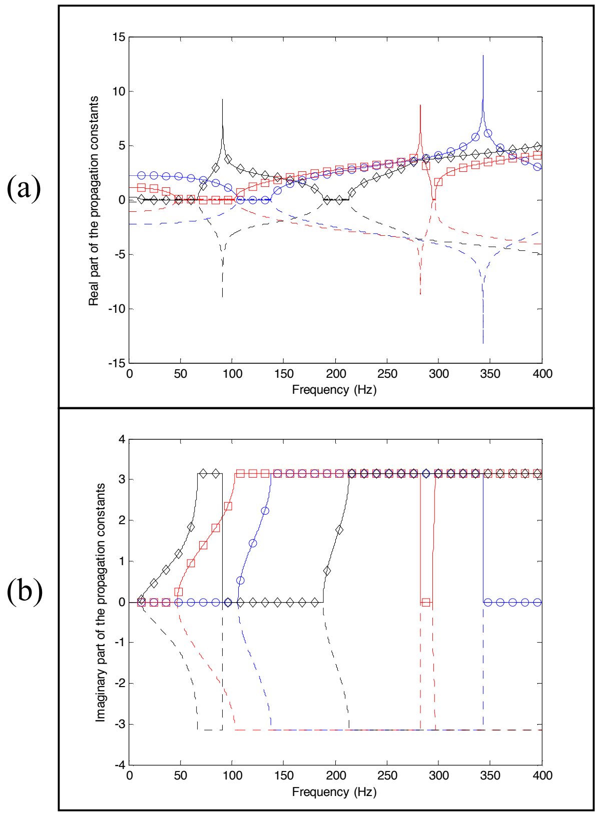


Fig.4 (a) Real and (b) imaginary parts of the propagation constants for a beam with three springs on each side. -◊-, constant 1; -□-, constant 2; -○-, constant 3.

Two types of zones can be distinguished: the propagation zones (pass-bands) and the attenuation zones (stop-bands). In case of undamped couplings, the propagation zones are defined by a purely imaginary propagation constant $\mu = -j\beta$. According to the Floquet-Bloch's theorem, crossing a single element is equivalent to multiply by $e^{-j\beta}$, which means without change of amplitude (propagating wave phenomenon) and with a phase difference. On the other hand, the attenuation zones are defined by a real propagation constant $\mu = -\alpha$. In that case, the amplitude of the wave will be multiplied by $e^{-\alpha}$ when travelling through a single element. The wave is dramatically attenuated and all the coupling points are vibrating in phase. With regard to the whole periodic structure, the more a wave will cross single elements, the more its amplitude will decrease, until becoming insignificant and having no influence on the rest of the system (evanescent wave phenomenon).

On Fig. 4, the first propagation constant shows two propagation zones, between 12 and 67 Hz and between 188 and 213 Hz. The second propagation constant also has two propagation zones, between 47 and 103 Hz and between 294 Hz and 297 Hz. The third propagation constant has only one propagation zone in the frequency domain of interest, from 106 to 138 Hz.

Note that the natural modes of the whole system will occur in propagation zones due to the necessity of transmitting the energy in all the structure.

3.4 Transfer matrix eigenvectors

According to Eq.(2), the eigenvectors are composed of displacements and forces. They represent the dynamic behaviour of the coupling points of the single element (that means the ends of the springs). Each eigenvector corresponds to a wave travelling into the structure. The displacements and the forces of these eigenvectors are respectively called eigendisplacements and eigenforces.

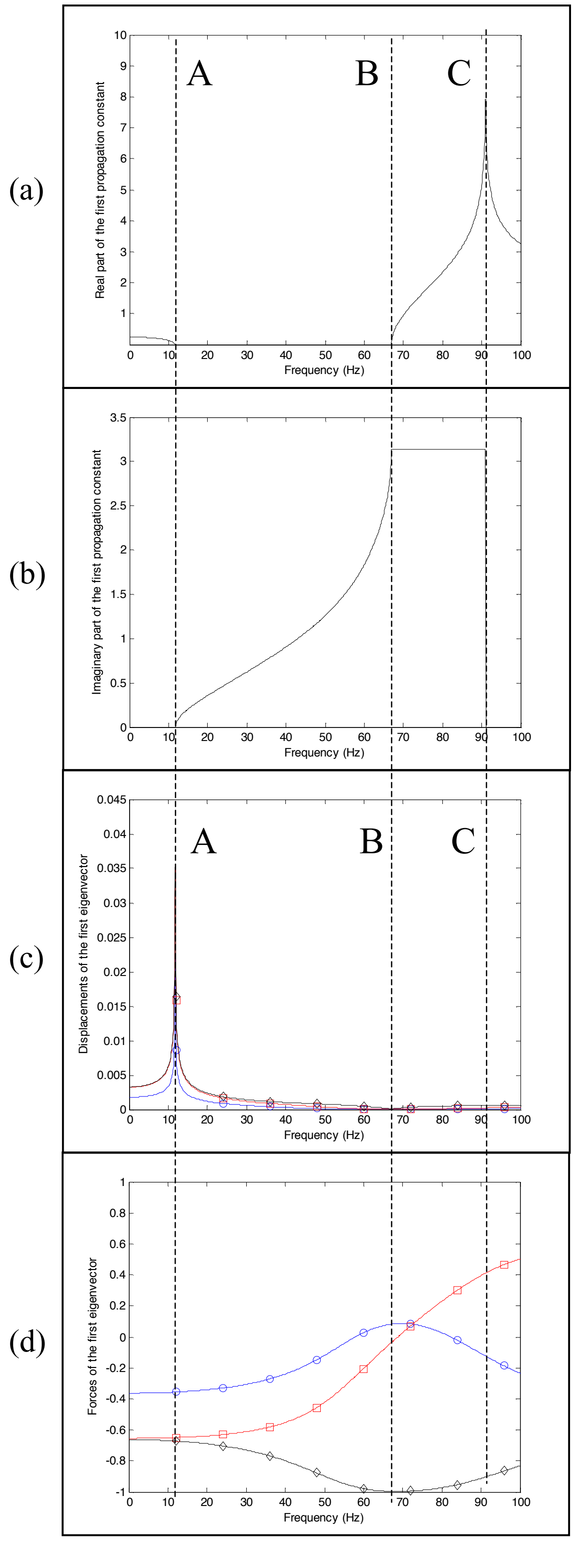
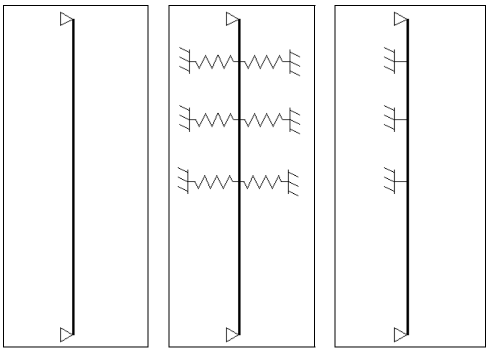


Fig.5 (a) Real and (b) imaginary parts of the first propagation constant ; (c) displacements and (d) forces of the first eigenvector. -◊-, coupling 1; -□-, coupling 2; -○-, coupling 3.

Fig. 5 shows simultaneously the first propagation constant (i.e. first eigenvalue) and the first eigenvector. The observations described here are also valid for the other eigenvectors. As shown by the real and imaginary parts of the propagation constant (Fig. 5a and 5b), the first propagation zone is comprised between the lines A and B (from 12 to 67 Hz). Concerning the eigendisplacements (Fig. 5c) and the eigenvectors (Fig. 5d), each curve corresponds to the end of one of the three couplings of the single element.

On the basis of the receptance matrix terms, Mead [3] showed that the natural frequencies of a single element with its ends free and fixed correspond to the bounding frequencies of the propagation zones. This feature can also be revealed from the observation of the transfer matrix eigenvectors. Fig. 5c shows significant eigendisplacements at the beginning of the propagation zone (line A), which points out an ability to move without constraint for the ends of the springs. So this frequency corresponds to a natural mode of a single element with its ends free, which is equivalent to a simply supported beam in the present case (Fig. 6a). On the contrary, the eigendisplacements are equal to zero at the end of the zone (line B), which is equivalent to block the ends of the coupling springs. This configuration being identical to a fixed single element (Fig. 6b), its natural frequencies constitute the upper bounds of the propagation zones.

The propagation zones can be interpreted as a continuous variation of the boundary conditions of the single element between the free (Fig. 6a) and the fixed case (Fig. 6b).



(a) (b) (c)

Fig.6 (a) Free single element; (b) Fixed single element; (c) Beam fixed at the position of the couplings.

The geometry of the single element under study reveals a third phenomenon concerning the propagation constants. At some frequencies (line C at 91 Hz), the real part tends to infinite, which corresponds to the complete wave attenuation. These frequencies have been identified as the natural frequencies of a beam blocked at the location of the couplings (Fig. 6c). With these boundary conditions, the springs can not transmit any displacement, which is equivalent to isolate the excited beam from the rest of the assembly.

4 Conclusion

The principles of the vibratory behaviour of a finned coil heat exchanger are studied through a simpler structure, an infinite periodic assembly of beams multi-coupled by springs. The whole periodic system is obtained from the repetition of a single element, which has been modelled by the transfer matrix formulation. The waves propagation into the structure is then studied through the eigenvalues (i.e. propagation constants) and vectors of the transfer matrix. The propagation constants show the propagation and the attenuation zones while the eigenvectors enable to get a physical meaning of the coupling points behaviour. Thanks to this periodic approach, the vibratory behaviour of the whole assembly of beams can be described only from the knowledge of the single element.

In this paper, the structure is infinite, allowing to apply straight the Floquet-Bloch's theorem to model the waves propagation into the system. In the next step, a more realistic finite structure will be studied. Thus the vibratory behaviour of the structure boundaries will have to be considered. It will be then necessary to adapt the formulation which actually relates the displacements of two coupling points, to take into account the reflected waves.

References

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