Conservative initial-boundary value problems for the Wide-Angle PE in waveguides with variable bottoms

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We consider the third-order, wide-angle Parabolic Equation (PE) in the context of Underwater Acoustics in a cylindrically symmetric medium consisting of water over a soft bottom of variable geometry. The initial-boundary-value problem for this equation with just a homogeneous Dirichlet bottom boundary condition may not be well-posed, for example when the bottom is downsloping. In previous work we proposed an additional boundary condition that, together with the zero-field condition on the bottom, yields a well-posed problem. Here, we continue our investigation of additional bottom boundary conditions that yield well-posed, physically correct problems. Motivated by the fact that the solution of the wide-angle PE in a domain with horizontal layers conserves its L^2 norm in the absence of attenuation, we identify additional boundary conditions that yield L^2-conservative solutions of the problem. After a range-dependent change of the depth variable that makes the bottom horizontal, we discretize the continuous problems by Crank-Nicolson type finite difference schemes, and show, by means of numerical experiment, that some of the new models yield accurate simulations of the acoustic field in standard, wedge-type domains with upsloping and downsloping bottoms.

1 Introduction

We consider the third-order, Claerbout-type wide-angle parabolic equation (PE) of underwater acoustics in a cylindrically symmetric medium

$$\big[1 + q\beta + \frac{q}{k_0^2}\partial_z^2\big]v_r = \mathrm{i}(p-q)k_0\big[\frac{1}{k_0^2}v_{zz} + \beta v\big] \qquad (1)$$

for $(z,r) \in [0,s(r)]\times[0,T]$, where $v = v(z,r)$, a complex-valued function of the depth z and range r, is the acoustic field generated by a time-harmonic point source of frequency f, k_0 is a reference wave number, p, q are complex constants such that $p = q + 1/2$ and $\beta(z,r)$ is a complex-valued function. The problem is posed on a single-layer domain with a variable bottom described by the positive smooth function $z = s(r)$. We supplement (1) by an initial condition modelling the sound source at $r = 0$, and a pressure-release boundary condition on the surface $z = 0$, *i.e.* we require that

$$\begin{aligned} &v(z,0) = v_0(z), \quad 0 \le z \le s(0), \\ &v(0,r) = 0, \quad 0 \le r \le T, \end{aligned} \qquad (2)$$

where v_0 is a given function. We also supplement (1) by the homogeneous Dirichlet bottom boundary condition

$$v(s(r),r) = 0, \quad 0 \le r \le T. \qquad (3)$$

There is a simple numerical and theoretical evidence (see for instance in Refs. [2], [3], [4]) that the initial-boundary-value problem (ibvp) (1)–(3) may not be well-posed, for example if the bottom is downsloping. In Refs. [2] and [3] an additional boundary condition was proposed that together with (1)–(3) yields a well-posed problem. In this note, motivated by the fact that solutions of (1)–(3) conserve the L^2 norm, *i.e.* satisfy

$$\|v(\cdot,r)\| = \|v_0\| \quad \text{for } r \ge 0,$$

where $\|v(\cdot,r)\| := (\int_0^{s(r)} |v(z,r)|^2\mathrm{d}z)^{1/2}$, in domains with a horizontal bottom when β and q are real, we seek additional boundary conditions that will render the problem well-posed and L^2-conservative (when β and q are real) even in the presence of variable bottom. We prove that the problem (1)–(3) is L^2-conservative if and only if the additional bottom boundary condition is chosen such that

$$\mathrm{Im}\big\{g_z(s(r),r)\,\overline{g(s(r),r)}\big\} = 0, \quad 0 \le r \le T, \qquad (4)$$

where the function g is defined by

$$g := qv_r - \mathrm{i}(p-q)k_0 v.$$

We then choose some specific boundary conditions that satisfy (4) and solve numerically the resulting ibvp by a finite difference scheme, after a range-dependent change of the depth variable that renders the bottom horizontal. We study the accuracy, stability, and conservation properties of this scheme in a series of numerical experiments and use it to simulate the acoustic field in standard wedge-type domains with upsloping and downsloping bottoms.

2 L^2 and H^1 estimates for various bottom boundary conditions

In the sequel we shall call an ibvp (*e.g.* the ibvp (1)–(3)) X-stable (where X is a normed linear space of functions of z) if $\|v(\cdot,r)\|_X \le C\|v_0\|_X$ for $0 \le r \le T$ and some constant C independent of v_0. In the upsloping case we have

Proposition 2.1 *If β, q are real, and $\dot{s}(r) \le 0$, $r \in [0,T]$, then the ibvp (1), (2), (4) is L^2-stable.*

For general bottom profiles, we have

Proposition 2.2 *If β and q are real, then the ibvp (1)–(3) is L^2-conservative, if and only if (4) holds.*

First, observe that (1) may be written as

$$v_r + \beta g + \frac{1}{k_0^2}g_{zz} = 0. \qquad (5)$$

The proof of the results given in Propositions 2.1 and 2.2 can be easily achieved by multiplying (5) with $\overline{g(z,r)}$ and integrating over $0 \le z \le s(r)$. Integrating by parts, taking imaginary parts, and using the fact that $g(0,r) = 0$ we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}r}\Big(\int_0^{s(r)} |v(z,r)|^2\mathrm{d}z\Big) = \dot{s}(r)\,|v(s(r),r)|^2 - \frac{2}{(p-q)k_0^3}\mathrm{Im}\big\{g_z(s(r),r)\,\overline{g(s(r),r)}\big\},$$

where a dot denotes differentiation with respect to r. Hence, using (3),

$$\|v(\cdot,r)\|^2 - \|v_0\|^2 = -\frac{2}{(p-q)k_0^3}\int_0^{s(r)} \mathrm{Im}\big\{g_z(s(\tau),\tau)\,\overline{g(s(\tau),\tau)}\big\}\mathrm{d}\tau, \qquad (6)$$

from which the propositions follow.

The following result concerns the H^1-stability of the problem (1)–(2) under various bottom boundary conditions.

Proposition 2.3 *The solution of the problem (1)–(2) is H^1-stable provided the following holds:*

(i) *For* $0 \le r \le T$, $\dot{s}(r)|v_z(s(r),r)|^2 \le 0$ *and*

$$\mathrm{Re}\{g_z(s(r),r)\,\overline{v(s(r),r)}\} = 0.$$

(ii) β, $q \neq 0$ *are real, and* $k_0 \max_{r\in[0,T]} s(r)$ *is sufficiently small.*

In order to prove this result we multiply (1) by $\overline{v(z,r)}$ and integrate over $0 \le z \le s(r)$. Taking real parts we see that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}r}\Big(\int_0^{s(r)} \big[\,|v_z|^2 - \frac{k_0^2}{q}(1+q\beta)\,|v|^2\big]\mathrm{d}z\Big) = \\
&-\frac{k_0^2}{q}\int_0^{s(r)} \partial_r(1+q\beta)\,|v|^2\mathrm{d}z + \dot{s}(r)|v_z(s(r),r)|^2 \\
&-\dot{s}(r)\frac{k_0^2}{q}\big(1+q\beta(s(r),r)\big)\,|v(s(r),r)|^2 \\
&+\frac{2}{q}\mathrm{Re}\{g_z(s(r),r)\,\overline{v(s(r),r)}\}.
\end{aligned} \tag{7}$$

Using now Poincaré's inequality, integrating over r and using the fact that

$$1 - C\Big(\frac{k_0^2}{q}\max_{z,r}|1+q\beta|\Big) > 0$$

for some positive constant C (this follows from *(ii)*), we arrive at the H^1 estimate

$$\|v_z(\cdot,r)\| \le C'\|v_z(\cdot,0)\|.$$

Under the hypotheses of Proposition 2.3 we conclude therefore that

1. The ibvp (1)–(3) is H^1-stable, if the bottom is upsloping ($\dot{s} \le 0$).
2. We observe that (3) and the condition $g(s(r),r) = 0$ gives $v_r(s(r),r) = 0$, hence
   $$v_z(s(r),r) = 0.$$
   Thus the ibvp (1)–(3) with the extra condition $g(s(r),r) = 0$ is L^2-conservative and H^1-stable for any bottom profile.
3. The ibvp (1), (2) with the bottom boundary condition $g_z(s(r),r) = 0$ is L^2- and H^1-stable, provided the bottom is upsloping ($\dot{s} \le 0$).

3 Finite difference schemes

If we perform the transformation $y = z/s(r)$, $t = r$, $u(y,t) := v(z,r)$ we see that (1) becomes

$$\begin{aligned}
&u_t - y\frac{\dot{s}(t)}{s(t)}u_y + c(y,t)\Big\{q\Big[u_t - y\frac{\dot{s}(t)}{s(t)}u_y\Big] - \mathrm{i}(p-q)k_0u\Big\} \\
&+\frac{1}{k_0^2s^2(t)}\partial_y^2\Big\{q\Big[u_t - y\frac{\dot{s}(t)}{s(t)}u_y\Big] - \mathrm{i}(p-q)k_0u\Big\} = 0
\end{aligned} \tag{8}$$

for $(y,t) \in [0,1]\times[0,T]$, where $c(y,t) := \beta(ys(r),r)$. We let $a(t) = \dot{s}(t)/s(t)$, $d(t) = 1/(k_0s(t))^2$,

$$f := q\Big[u_t - y\frac{\dot{s}(t)}{s(t)}u_y\Big] - \mathrm{i}(p-q)k_0u,$$

and consider the following ibvp for (8)

$$\begin{aligned}
&u_t - ya(t)u_y + c(y,t)f + d(t)f_{yy} = F(y,t), \\
&0 \le t \le T, \quad 0 \le y \le 1, \\
&u(y,0) = u_0(y) = v_0(ys(0)), \quad 0 \le y \le 1, \\
&u(0,t) = u(1,t) = 0, \quad 0 \le t \le T,
\end{aligned} \tag{9}$$

where we added a nonhomogeneous term, $F(y,t)$, for greater generality. We append to (9) an extra bottom boundary condition that will imply

$$\mathrm{Im}\big\{f_y(1,t)\,\overline{f(1,t)}\big\} = 0, \quad 0 \le t \le T. \tag{10}$$

Observe that (10) is the form that (4) takes after the horizontal transformation. By Proposition 2.1, if $c(y,t)$ and q are real, and $F = 0$ the ibvp (9)–(10) is L^2-conservative in the z, r variables, *i.e.* in the sense that

$$\sqrt{s(t)}\,\|u(\cdot,t)\| = \sqrt{s(0)}\,\|u_0\|,$$

where $\|\cdot\|$ denotes now the L^2 norm on $[0,1]$.

There are many boundary conditions that imply (10), for example

$$f_y(1,t) = \lambda(t)f(1,t) \quad \text{or} \quad f(1,t) = \mu(t)f_y(1,t)$$

where λ, μ are real functions. The second class includes the simple condition $f(1,t) = 0$. We chose to study numerically the ibvp

$$\begin{aligned}
&u_t - ya(t)u_y + c(y,t)f + d(t)f_{yy} = F(y,t), \\
&0 \le t \le T, \quad 0 \le y \le 1, \\
&u(y,0) = u_0(y), \quad 0 \le y \le 1, \\
&u(0,t) = u(1,t) = 0, \quad 0 \le t \le T, \\
&f(1,t) = \gamma(t)\dot{s}(t)f_y(1,t) + m(t), \quad 0 \le t \le T,
\end{aligned} \tag{11}$$

where $\gamma(t)$ is a real and $m(t)$ a complex function. Note that we added a nonhomogeneous term $m(t)$ for greater generality and use the term $\dot{s}$ in $\mu = \gamma\dot{s}$ in order to study the influence of the bottom geometry in the computations. In what follows we will define a simple finite difference scheme of Crank-Nicolson type for (11) and solve the problem numerically in artificial and realistic domains.

We define a uniform partition of the interval $[0,T]$ with step $k := T/N$, nodes $t^n := nk$, $n = 0,\ldots,N$, and intermediate nodes $t^* := t^n + \frac{k}{2}$, $n = 0,\ldots,N-1$. We let also $h := 1/(J+1)$ and consider a uniform partition of the interval $[0,1]$ with nodes $y_j := jh$, $j = 0,\ldots,J+1$. We let U_j^n be the approximation to $u(y_j,t^n)$ defined by the scheme:

Step 1. Set $U_j^0 = u_0(y_j)$ for $1 \le j \le J$, $U_0^0 = U_{J+1}^0 = 0$.

Step 2. For $n = 1,\ldots,N-1$, compute U_j^{n+1}, $0 \le j \le J+1$, such that

$$\begin{aligned}
&\frac{U_j^{n+1} - U_j^n}{k} - y_j\,a(t^*)\frac{U_{j+1}^{n+\frac12} - U_{j-1}^{n+\frac12}}{2h} + c(y_j,t^*)f_j^n \\
&\qquad +d(t^*)\Delta f_j^n = F(y_j,t^*), \quad 1 \le j \le J, \\
&U_0^{n+1} = U_{J+1}^{n+1} = 0,
\end{aligned}$$

where $U_j^{n+\frac{1}{2}} = (U_j^n + U_j^{n+1})/2$ for $j = 0, \dots, J+1$, and

$$\Delta f_j^n := \frac{f_{j+1}^n - 2f_j^n + f_{j-1}^n}{h^2}, \quad 1 \le j \le J-1,$$

$$\Delta f_J^n := \frac{c_1 f_{J-1}^n - c_2 f_J^n}{h^2} - \frac{2m(t^*)}{h(3\gamma(t^*)\dot{s}(t^*) - 2h)},$$

with c_1 and c_2 defined by

$$c_1 = \frac{2(\gamma(t^*)\dot{s}(t^*) - h)}{3\gamma(t^*)\dot{s}(t^*) - 2h}, \quad c_2 = \frac{2(\gamma(t^*)\dot{s}(t^*) - 2h)}{3\gamma(t^*)\dot{s}(t^*) - 2h},$$

and where

$$f_j^n := q\Big(\frac{U_j^{n+1} - U_j^n}{k} - y_j \frac{\dot{s}(t^*)}{s(t^*)} \big(\frac{U_{j+1}^{n+\frac{1}{2}} - U_{j-1}^{n+\frac{1}{2}}}{2h}\big)\Big)$$

$$-\mathrm{i}(p-q)k_0 U_j^{n+\frac{1}{2}}, \quad 1 \le j \le J,$$

and $f_0^n := 0$. This scheme requires solving a pentadiagonal system of linear equations for each n, and can be proved to be L^2-stable and of order of accuracy $O(h^2 + k^2)$ when q, c are real, and $\gamma(t) = m(t) = 0$ (*i.e.* in the case $f(1,t) = 0$).

4 Numerical experiments

We considered the problem (11) with $k_0 = 0.1$, $q = 1/4$, $p = 3/4$, and $c(y,t) = 1 + y$. We took as its exact solution the function $u_{\mathrm{ex}}(y,t) = y^2(y-1)\exp(2t)$ for a suitable F and $u_0(y) = u_{\mathrm{ex}}(y,0)$. The exact solution $u_{\mathrm{ex}}(y,t)$ satisfies $f(1,t) = (\dot{s})^2 + m(t)$ for suitable m (*i.e.* $\gamma(t) = \dot{s}(t)$), and $u_{\mathrm{ex}} = 0$ at $y = 0$ and $y = 1$. We ran our code for the following bottom cases on $[0,T]$:

- Case 1: $s(t) = -t + 2$. $\dot{s}(t) = -1 < 0$.
- Case 2: $s(t) = t + 2$. $\dot{s}(t) = 1 > 0$.
- Case 3: $s(t) = \exp(-t)$. $\dot{s}(t) = -\exp(-t) < 0$.
- Case 4: $s(t) = \exp(t)$. $\dot{s}(t) = \exp(t) > 0$.
- Case 5: $s(t) = t^2 + 2t + 1$. $\dot{s}(t) = 2t + 2 > 0$.
- Case 6: $s(t) = \cos(2\pi t) + 2$. $\dot{s}(t) = -(2\pi)\sin(2\pi t)$. Thus, $\dot{s}(t) < 0$ for $0 < t < 0.5$, and $\dot{s}(t) > 0$ for $0.5 < t < 1$.

For all cases we obtained experimentally second-order accuracy $O(k^2 + h^2)$ in the discrete L^2-norm $\|U^n\| := (h\sum_{j=1}^J |U_j^n|^2)^{1/2}$, and in order to check the degree of conservativity of the numerical scheme, we computed in each case the discrete weighted L^2 norm $\|U^n\|_* = (s(t^n)h\sum_{j=1}^J |U_j^n|^2)^{1/2}$ in the case of the homogeneous problem ($F = m = 0$) with the same initial data as above. The results for $N = J = 1280$ appear in Fig. 1. We observe that the finite difference scheme is practically L^2-conservative in the cases 2, 4, and 5 (*i.e.* in the downsloping cases) and loses conservativity in the upsloping cases 1 and 3. This behavior may also be seen in the periodic case 6.

We also ran our code in the case of the standard ASA wedge test case [5] and $m = F = 0$ and for various functions $\gamma(t)$, taking the complex Padé coefficient $q = (0.252252311, -1.35135138\mathrm{e}-02)$. A normal-mode

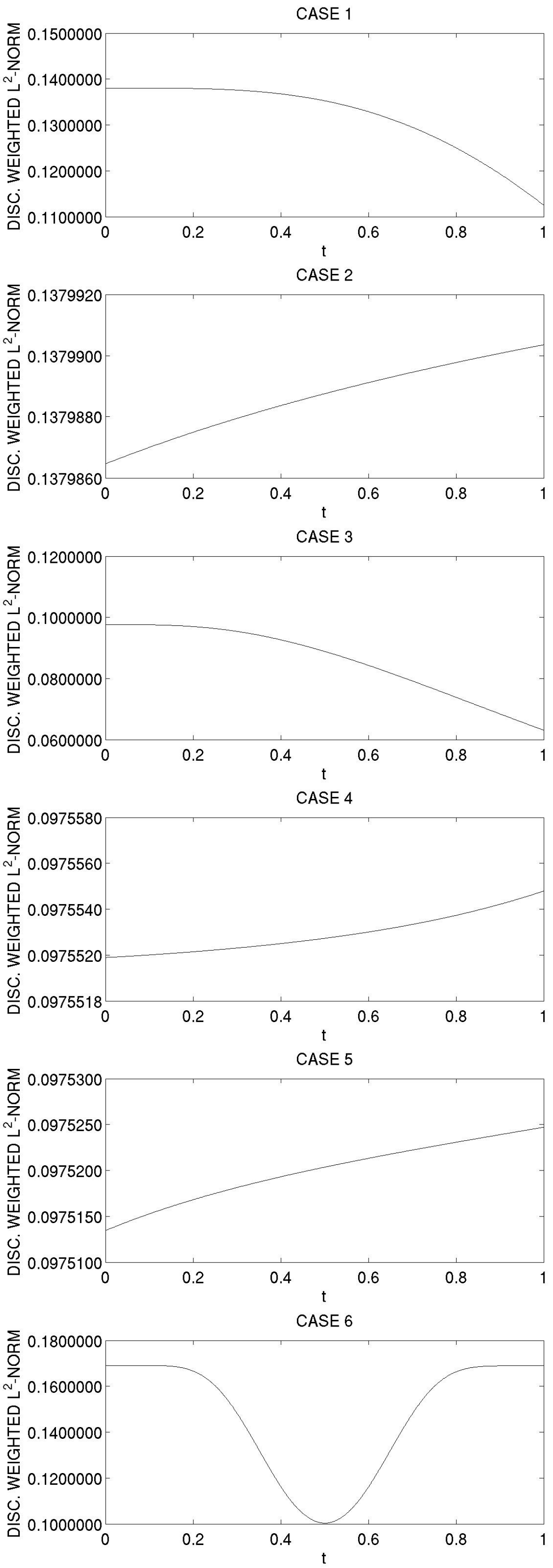

Figure 1: Discrete weighted L^2-norm $\|U^n\|_*$ for various bottom cases as a function of t^n - Problem (11) with $\gamma = \dot{s}$, $F = m = 0$.

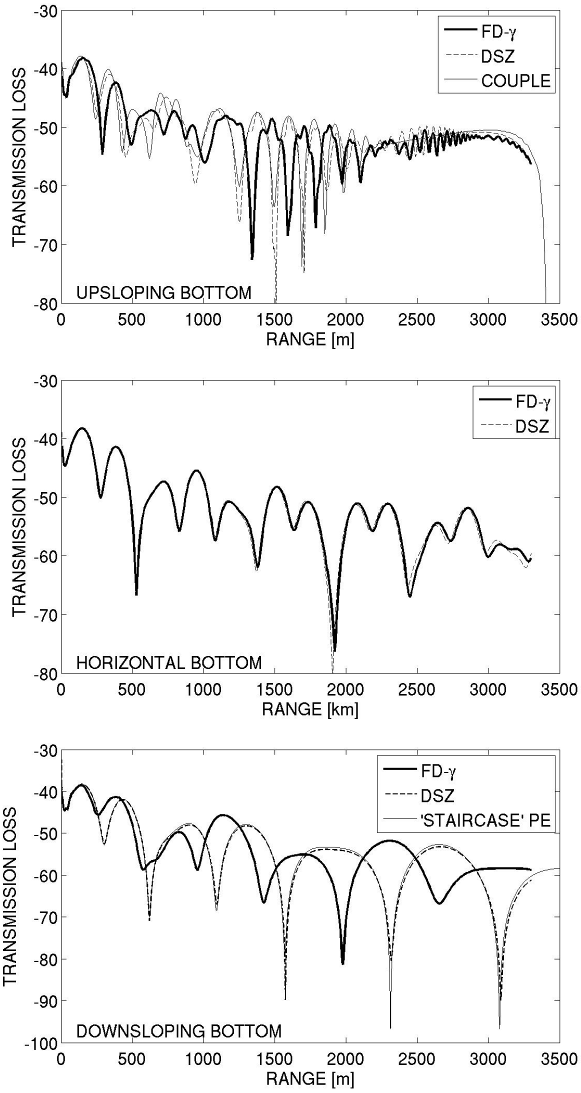

Figure 2: Transmission loss at a receiver depth of 30 m when $\gamma(t) = 1$.

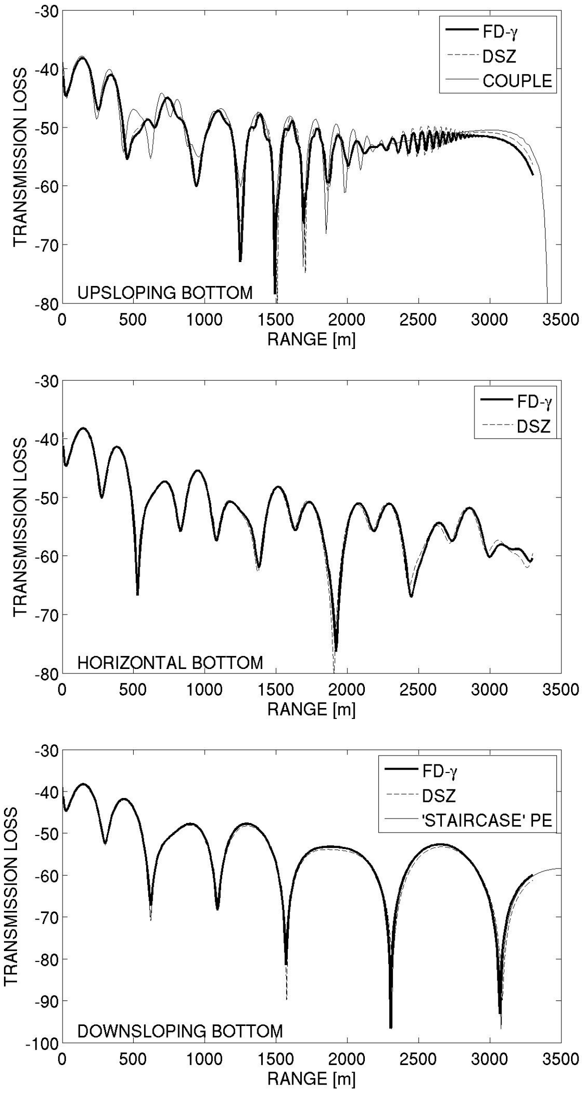

Figure 3: Transmission loss at a receiver depth of 30 m when $\gamma(t) = \dot{s}(t)$.

starter (6 modes) source (*cf.* [2], [3]) of frequency $f = 25$ Hz was placed at a depth of 100 m. We took the sound speed constant and equal to 1500 m/sec and no attenuation (thus, $\mathrm{Im}(\beta) = 0$), calculated the transmission loss ($-20\log_{10}(|v(z,r)|/\sqrt{r})$ at a receiver depth of 30 m, and compared our results to which we will refer to as FD-γ for better clarity, with the results of the code implementing the same problem with the extra bottom condition

$$u_{yy}(1,t) = 2ik_0 s(t)\dot{s}(t)u_y(1,t), \quad t \geq 0,$$

of Refs. [2] and [3] (called in the sequel DSZ condition) for the following three bottom profiles (all distances expressed in meters):

$$s(r) = 200(1 - r/4000), \quad \text{(upsloping)}$$
$$s(r) = 200, \quad \text{(horizontal)}$$
$$s(r) = 200(1 + r/4000), \quad \text{(downsloping)}$$

taking, in each case, $\gamma(t) = 1$, $\gamma(t) = \dot{s}(t)$ and $\gamma(t) = 0$. The results are shown in Figs. 2–4 in the z and r variables. Note that in the upsloping cases, we also compare our results with those of COUPLE [6], and in the downsloping cases with those of a wide-angle parabolic equation based code with staircase bottom discretization.

In the case $\gamma(t) = 1$ (Fig. 2) we have good agreement only in the horizontal bottom case, *i.e.* when $f = 0$. In the sloping bottom cases the results do not agree. In the case $\gamma(t) = \dot{s}(t)$ (Fig. 3), the results are much better (in this case the coefficient $\gamma(t)\dot{s}(t) = \dot{s}^2(t)$ is very small) and are practically the same with the results corresponding to $\gamma(t) = 0$ (Fig. 4). Notice that when $\gamma(t) = 0$ the extra boundary condition in (11) becomes $f(1,t) = 0$ which implies that $u_y(1,t) = 0$ (or, equivalently, $v_r(s(r), r) = 0$ in the variable domain) when we take into account $u(1,t) = 0$. We conclude therefore that when the term $\gamma(t)\dot{s}(t)$ is small, the present model gives accurate simulations of the sound field in these variable bottom domains.

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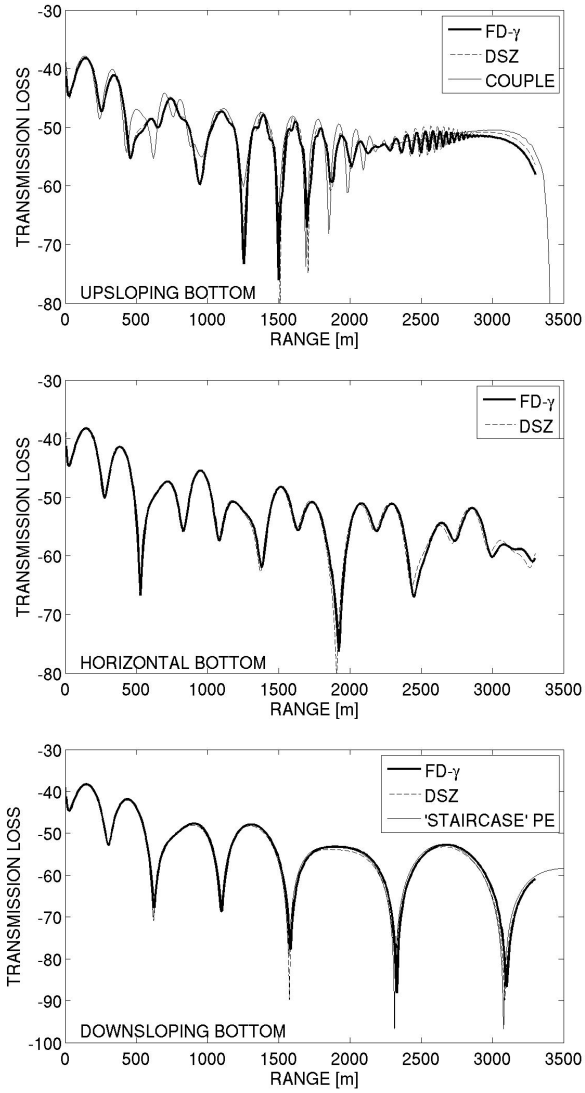

Figure 4: Transmission loss at a receiver depth of 30 m when $\gamma(t) = 0$.

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