Coupled wavenumbers of structural acoustic waveguides: a unified asymptotic approach

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In the literature, the coupled wavenumbers in flexible-walled acoustic waveguides have been found mainly using numerical methods for a fixed set of parameters. These solutions, although useful, do not continuously track the coupled wavenumbers as the fluid-loading parameter is varied from small to large values. Such a continuous tracking is possible by applying the asymptotic methods to the coupled dispersion relation. Analytical formulae for the coupled wavenumbers can also be found. In this work, we present a consistent, unified and physically insightful view of structural acoustic coupling in the context of a planar structural-acoustic waveguide (with two different BCs), the axisymmetric and beam modes of a fluid-filled circular cylindrical shell and finally a fluid-filled elliptic cylindrical shell. In all the five cases, we use a single fluid-structure-coupling asymptotic parameter (μ). The regular perturbation method is used to solve the coupled dispersion relation for small and large values of μ. The circular and the elliptic cylinders necessitate the use of additional asymptotic parameters. A general trend in all systems is that a given wavenumber branch transits from a rigid-walled solution to a pressure-release solution with increasing μ. Also, the wavenumber curves veer where the earlier uncoupled wavenumbers intersected.

1 Introduction

A significant amount of work has been carried out on the dispersion characteristics of structural-acoustic waveguides such as fluid-filled cylindrical shells and rectangular waveguides. Earlier workers have found solutions to these systems using numerical techniques and have discussed various physical implications of the solutions (such as mass or stiffness loading of one medium over the other). There is now enough literature available that a unified presentation seems required which can highlight the common features of all such waveguide systems. In the present article we seek to achieve this goal of presenting the common characteristics of structural-acoustic waveguides using the method of asymptotics.

Usage of asymptotic methods in structural acoustics is common, though not widespread ([1, 2] to name a few). In asymptotic methods, we seek an approximate analytical solution to an otherwise analytically intractable problem (for example, transcendental equations, nonlinear differential equations, *etc.*). The problem though intractable, is *in some way close* to an analytically solvable one. Mathematically, the problem involves a parameter (called the asymptotic parameter) which if set to zero can lead to closed form solutions. The reduced problem obtained by setting the asymptotic parameter to zero is called the unperturbed problem. We thus seek solutions in the form of perturbations to the solutions of the unperturbed problem. The solution so obtained is increasingly accurate as the asymptotic parameter approaches zero (see [3] for further discussion). Physically, the method reveals the phenomenon involved as a correction over a better-understood simpler process. Further, the first order correction term captures the main difference between the two problems.

Using asymptotic methods, Crighton had established that the dynamics of an infinite plate submerged in an unbounded acoustic fluid can be approached as a correction to the well-known *in vacuo* plate dynamics (see [1] and the cross-references therein). Crighton's alternative asymptotic viewpoint on this subject was illuminating. We have extended Crighton's ideas in our recent works on structural acoustic waveguides [4, 5] and observed certain common features of the coupled dispersion characteristics for various configurations (shown in figure 1). We wish to present these in this article. Specifically, we show that under suitable conditions, the coupled wavenumbers of these systems (despite their geometrical differences) may be obtained as corrections to the simpler cases of (1) the *in vacuo* structural wavenumber, (2) the wavenumber of the rigid-walled acoustic cut-ons and (3) the wavenumber of the pressure-release acoustic cut-ons. As the geometry gets progressively complicated, we introduce asymptotic parameters which reduce the new problem to a correction over the previous one. For the geometries considered, this brings in an alternative unified asymptotic view-point which is physically insightful.

2 Two dimensional waveguide

As shown in figure (1a), a two dimensional structural acoustic waveguide consists of a flexible plate loaded with a finite fluid column. There are two boundary conditions applicable at the top surface of the fluid, namely:- (a) y-directional acoustic velocity $v_y(x,a)=0$, (b) acoustic pressure $p(x,a)=0$. The non-dimensional coupled dispersion equations for these cases are given by [4, 6]

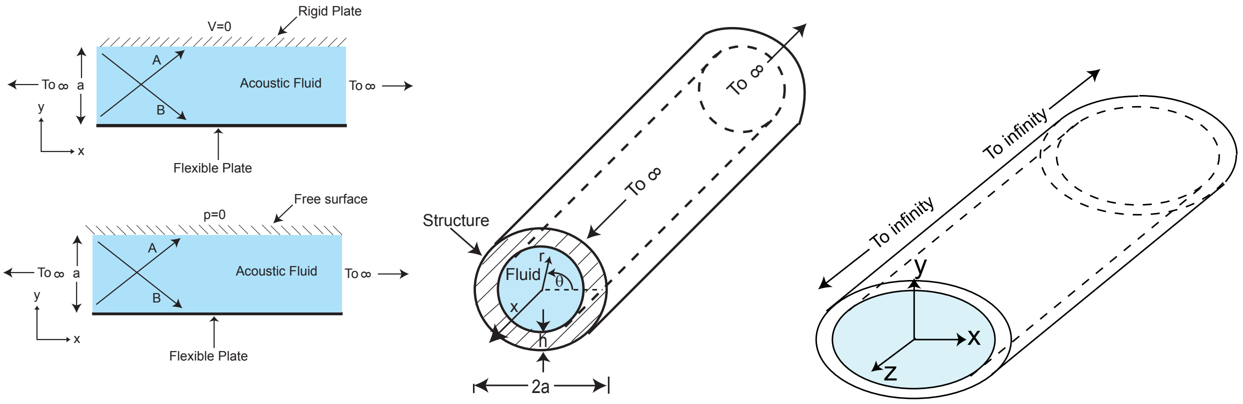
$$\left[\frac{\xi^4}{\Omega^2}-1\right]\left[\lambda\sqrt{\Omega^2-\xi^2}\tan\left(\lambda\sqrt{\Omega^2-\xi^2}\right)\right]+\mu=0, \quad (1a)$$

$$\left[\frac{\xi^4}{\Omega^2}-1\right]\left[\lambda\sqrt{\Omega^2-\xi^2}\cot\left(\lambda\sqrt{\Omega^2-\xi^2}\right)\right]-\mu=0, \quad (1b)$$

respectively.

Throughout the article, we shall denote the dimensional frequency by ω, the dimensional wavenumber by k_x, the fluid density by ρ_f. In the above equations, the non-dimensionalization has been done with respect to the coincidence conditions (ω_c is the coincidence frequency and k_c is the coincidence wavenumber). Thus, we have the non-dimensional frequency $\Omega=\omega/\omega_c$, the non-dimensional wavenumber $\xi=k/k_c$, the non-dimensional fluid loading parameter $\mu=\rho_f a/m$ (where m is the mass per unit area of the plate) and the non-dimensional fluid column height as $\lambda=k_c a$.

In the equations above, consider the condition $\mu=0$. In this case, $\xi=\sqrt{\Omega}$ is a solution, which corresponds to the *in vacuo* bending wavenumber of the plate. Further, for equation (1a), we get additional solutions of the form $\lambda\sqrt{\Omega^2-\xi^2}=n\pi$, $n\in 0,1,2,\ldots$. These wavenumbers branches correspond to the wavenumbers of the acoustic duct with symmetric wall conditions at both $y=0$ and $y=a$. Thus, the flexible plate behaves as a rigid wall in this case. Similarly, for equation (1b), we get solutions of the form $\lambda\sqrt{\Omega^2-\xi^2}=(2n+1)\pi/2$, $n\in 0,1,2,\ldots$. These wavenumbers branches correspond



(a) Two dimensional waveguide (b) Fluid-filled circular cylindrical shell (c) Fluid-filled elliptic cylindrical shell

Figure 1: Structural acoustic waveguide systems studied in this article.

to the wavenumbers of the acoustic duct with antisymmetric wall conditions at $y = 0$ and $y = a$. As, $y = a$, is the free surface with $p = 0$, we may infer that at $y = 0$, we have the rigid-wall condition. Thus, again the flexible plate behaves as a rigid wall. Now, with $0 < \mu \ll 1$, solution to the equations (1) will be perturbations to the wavenumber branches discussed above. Using the regular perturbation method, these solutions can be found. The basic μ-based asymptotic series holds good for all frequencies other than the coincidence frequencies (i.e., frequencies wherein the *in vacuo* structural wavenumber equals the wavenumber of the rigid walled acoustic duct). At the coincidence frequencies, alternative $\sqrt{\mu}$-based asymptotic expansions are found. All these expressions have been worked out in our previous works [4, 6].

For the other extreme value of $\mu = \infty$, we take the transformation $\mu' = 1/\mu$ and consider $\mu' = 0$. The coupled dispersion equation (1), rewritten in terms of μ', gives

$$\mu' \left[\frac{\xi^4}{\Omega^2} - 1\right] \lambda\sqrt{\Omega^2 - \xi^2} + \cot\left(\lambda\sqrt{\Omega^2 - \xi^2}\right) = 0, \qquad (2a)$$

$$\mu' \left[\frac{\xi^4}{\Omega^2} - 1\right] \lambda\sqrt{\Omega^2 - \xi^2} - \tan\left(\lambda\sqrt{\Omega^2 - \xi^2}\right) = 0. \qquad (2b)$$

Using similar arguments as presented earlier, we infer that with $0 < \mu' \ll 1$ (*viz.* large μ), the flexible plate behaves as a pressure release boundary. Analytical expressions for this case can be found using a regular perturbation method [4, 6]. The results for all the above cases are schematically presented in Figure (2).

In summary, the results obtained help us to continuously track the coupled wavenumber branches as the fluid-loading parameter μ changes from small to large values (indicated schematically in figure (2)). The approach indicates how the coupled wavenumbers give back the uncoupled wavenumbers for extreme values of μ.

3 Circular cylindrical shell

The dispersion equation of a circular cylindrical shell (of radius a, density ρ_s, shell thickness h, extensional wave speed c_L and Poisson's ratio ν) in the n^{th} circumferential mode is given by [7]

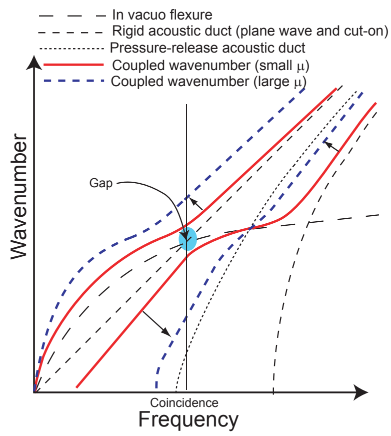


Figure 2: Schematic of the coupled wavenumber solutions. Arrows indicate transition of solutions from small μ to large μ.

$$\mathbf{L}\begin{Bmatrix} u_n \\ v_n \\ w_n \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}.$$

The elements of the matrix $\mathbf{L}$ are given by

$$L_{11} = -\Omega^2 + \kappa^2 + \frac{1-\nu}{2}n^2, \; L_{12} = L_{21} = \frac{1}{2}(1+\nu)n\kappa,$$

$$L_{13} = L_{31} = \nu\kappa, L_{22} = -\Omega^2 + \frac{1-\nu}{2}\kappa^2 + n^2, L_{23} = L_{32} = n^2.$$

The term L_{33} is given by

$$-\Omega^2 + 1 + \beta^2\left(\kappa^2 + n^2\right)^2 \text{ for } \textit{in vacuo}\,,$$

$$-\Omega^2 + 1 + \beta^2\left(\kappa^2 + n^2\right)^2 - \frac{\Omega^2}{\chi}\left(\frac{\rho_f a}{\rho_s h}\right)\left[\frac{J_n(\chi)}{J_n'(\chi)}\right] \text{ with fluid.}$$

In the above, $\chi = \sqrt{\Omega^2 \frac{c_L^2}{c_f^2} - \kappa^2}$ and u_n, v_n, w_n are the displacements in the longitudinal, circumferential and the radial directions, respectively. In this section, the non-dimensional frequency is given by $\Omega = \omega a / c_L$, the non-dimensional wavenumber is $\kappa = k_x a$ and $\beta^2 = h^2/12a^2$.

3.1 Axisymmetric mode

From the above equation, it is clear that for the axisymmetric mode (*viz.* $n = 0$), the circumferential vibra-

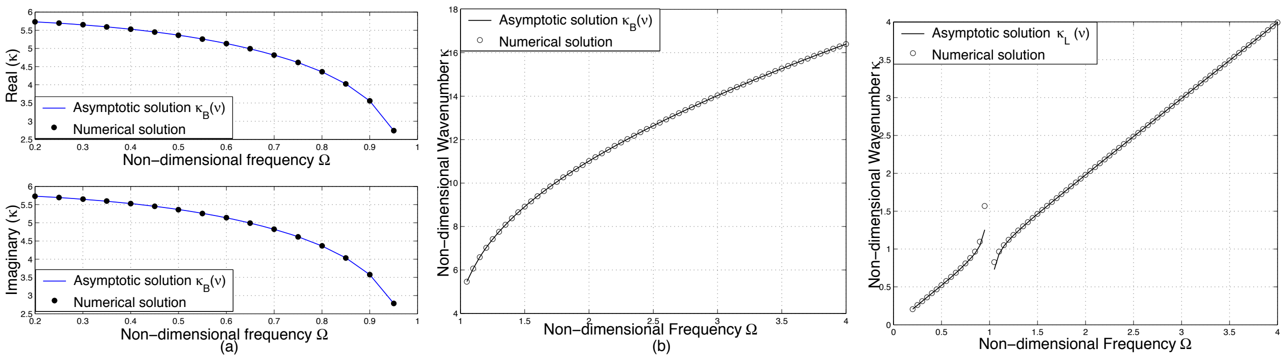


Figure 3: Wavenumbers of an *in vacuo* circular cylindrical shell with $\nu = 0.25$, $h/a = 0.05$. (a) Bending wavenumber below the ring frequency ($\Omega < 1$). (b) Bending wavenumber above the ring frequency ($\Omega > 1$). (c) Longitudinal wavenumber.

tion is uncoupled from the vibration in the other two directions. The radial and the longitudinal vibration though are coupled. Further, the coupling (given by the product of the off-diagonal terms L_{13} and L_{31}) depends directly on ν^2. For practical problems, we know that $0 < \nu^2 \ll 1$. This motivates an asymptotic series solution for the bending and longitudinal wavenumbers in terms of ν^2. These solutions have been found in our earlier work [5]. In figure (3), we present the comparison of the asymptotic solutions with the numerical solutions.

From equation (3), the coupled dispersion equation for the fluid-filled case is given by

$$|\mathbf{L}| = \overbrace{(-\Omega^2+\kappa^2)}^{L}\Big[\underbrace{(-\Omega^2+1+\beta^2\kappa^4)}_{B}\overbrace{J_1(\chi)}^{R}\underbrace{\chi}_{A} + \Omega^2\overbrace{\left(\frac{\rho_f a}{\rho_s h}\right)}^{\mu=}\overbrace{J_0(\chi)}^{P}\Big] - \underbrace{\nu^2\kappa^2 J_1(\chi)\chi}_{\text{Poisson's effect}} = 0. \quad (3)$$

From the equation above, it is clear that with $\nu = 0$ and $\mu = 0$, the solutions are the roots of the terms indicated as L, B, R and A. The physical significance of these roots is explained in Table (1). For $0 < \nu, \mu \ll 1$, a double asymptotic expansion (with both μ and ν as the asymptotic parameters) yields closed form expressions for the coupled wavenumbers. These have been worked out in our earlier paper [5]. Analogously, for small ν and large μ, using a transformation of variable to $\mu' = 1/\mu$, (thus μ' is small), the coupled wavenumbers are found as perturbations to the roots of P. In summary, we find that for small fluid-loading (characterized by the parameter μ) coupled wavenumbers are perturbations to the *in vacuo* flexural wavenumbers and the wavenumbers of the rigid acoustic duct. With increasing μ, these perturbations increase until for large μ, the coupled wavenumbers are better identified as perturbations to the wavenumbers of the pressure-release acoustic duct.

3.2 Beam Mode

For $n > 0$, the off-diagonal terms of the matrix $\mathbf{L}$ become non-zero. Thus, the axial, radial and circumferential vibrations are coupled. As noted by Fuller [8],

Term	Significance of roots
L	Longitudinal wavenumber
B	Bending wavenumber
R	Rigid duct wavenumber
A	Wavenumber of acoustic plane wave
P	Pressure-release duct wavenumber

Table 1: Significance of the roots of each term of the coupled dispersion equation (3).

these higher order modes resemble the $n = 1$ mode in a qualitative sense. Thus, we choose the $n = 1$ mode (*viz.* the beam mode) as the representative case for our study on higher order modes of the circular cylindrical shell.

The asymptotics of the beam-mode vibration of a circular cylindrical shell is done in two parts:- (1) high frequency $\Omega \gg 1$ (2) low frequency $\Omega \ll 1$. The longitudinal and torsional modes cut-on only beyond a cut-on frequency [8]. Thus, these wavenumbers represent propagating waves only for the high frequency. To determine these wavenumbers we rescale the frequency as $\Omega' = \epsilon\Omega$ and the wavenumber as $\kappa' = \epsilon\kappa$, where $0 < \epsilon \ll 1$ is a small fictitious rescaling parameter. Since the bending wavenumber varies as the square root of frequency, for finding this wavenumber we rescale the wavenumber as $\kappa' = \sqrt{\epsilon}\kappa$. After the rescalings are done, a regular perturbation method gives the asymptotic series solution for the wavenumbers. These results are presented in [9].

Analogously, for the low frequency calculations, we introduce the rescaled frequency variable $\Omega' = \Omega/\epsilon$. At low frequencies only the bending wave has a propagating nature. To determine this wavenumber, we introduce the rescaling $\kappa' = \kappa/\sqrt{\epsilon}$. A regular perturbation method on the rescaled variables yields closed form analytical expressions for the bending wavenumber. It can be noted from the perturbation solution that the one term asymptotic solution matches with the Euler-Bernoulli model while the two term asymptotic solution is in agreement with the Timoshenko model.

For both the high and low frequency ranges, the coupled dispersion equation is solved for small and large values of μ. The perturbation solution procedure is the same as described previously. The qualitative nature of the coupled wavenumbers found is also identical to that described previously for the axisymmetric mode.

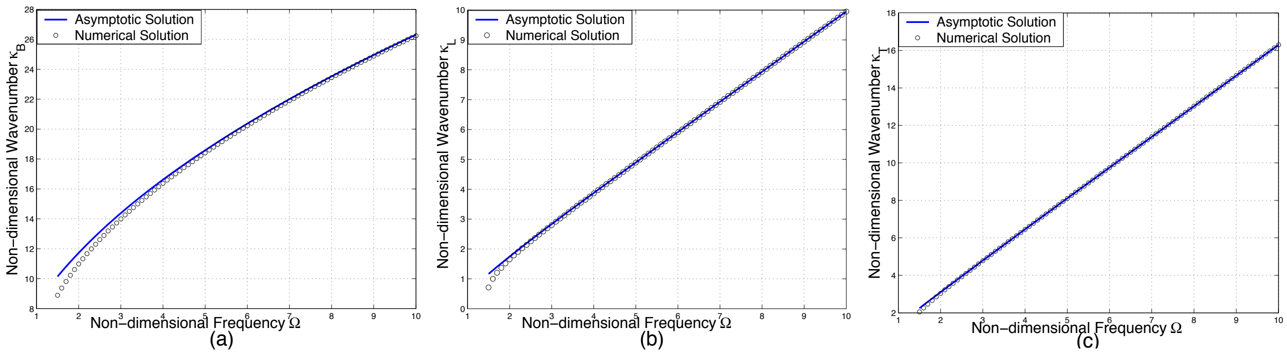


Figure 4: Wavenumbers of an *in vacuo* circular cylindrical shell with $\nu = 0.25$, $h/a = 0.05$ for high frequencies ($\Omega \gg 1$). (a) Bending wavenumber. (b) longitudinal wavenumber. (c) Torsional wavenumber.

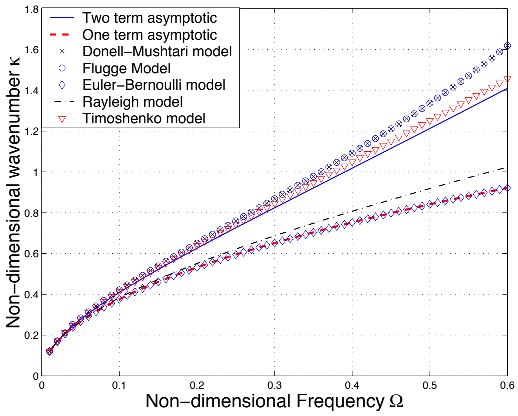


Figure 5: Bending wavenumber of an *in vacuo* circular cylindrical shell with $\nu = 0.25$, $h/a = 0.05$ for low frequencies ($\Omega \ll 1$).

4 Elliptic cylindrical shell

Initially, using the shallow shell theory [10], we study the dispersion characteristics of an *in vacuo* elliptic cylindrical shell for small values of the eccentricity (e). The use of shallow shell theory essentially limits the analysis to high frequencies only. Using e as an asymptotic parameter, the shell equations are simplified and finally the elliptic shell equations are formulated as a correction over that of the circular cylindrical shell [11].

Using ϵ as the asymptotic parameter (which depends on eccentricity (e)) and ϕ as an Airy-type stress function, the non-dimensional forms of the governing equations for the elliptic shell are given by

$$[1-\epsilon\cos(2\eta)]\left[\kappa^4\bar{w} - 2\kappa^2\frac{\partial^2\bar{w}}{\partial\eta^2} + \frac{\partial^4\bar{w}}{\partial\eta^4} - \frac{(1-\nu^2)}{\beta^2}\Omega^2\bar{w}\right] - \kappa^2\bar{\phi} = 0, \quad (4a)$$

$$[1-\epsilon\cos(2\eta)]\left[\kappa^4\bar{\phi} - 2\kappa^2\frac{\partial^2\bar{\phi}}{\partial\eta^2} + \frac{\partial^4\bar{\phi}}{\partial\eta^4}\right] + \frac{1-\nu^2}{\beta^2}\kappa^2\bar{w} = 0, \quad (4b)$$

where bar denotes non-dimensional quantities.

In this perturbation based approach, the objective is to understand the principal effect of eccentricity. As a novel contribution, using a symmetry-based classification for the modes and a harmonic balance technique, the *in vacuo* structural wavenumbers for the elliptic geometry are found as a correction over that of the circular cylindrical geometry. Acoustic analysis of elliptic ducts is known in the literature [12].

Next, as in the previous geometries, the coupled equations are formulated in terms of a fluid-loading parameter (μ) representing the ratio of masses of the fluid and the structure per unit area of the curved shell surface. In particular, the first equation is modified as

$$\left[\kappa^4\bar{w} - 2\kappa^2\frac{\partial^2\bar{w}}{\partial\eta^2} + \frac{\partial^4\bar{w}}{\partial\eta^4} - \frac{1-\nu^2}{\beta^2}\Omega^2\bar{w} - \frac{1-\nu^2}{\beta^2}\Omega^2\mu\left(\frac{Ce_m(\xi_0,q)}{Ce'_m(\xi_0,q)}\right)\bar{w}\right](1-\epsilon\cos(2\eta)) - \kappa^2\bar{\phi} = 0,$$

while the second equation remains unchanged.

Using qualitative arguments, we find that for extreme values of μ, the relation of coupled wavenumbers to the uncoupled structural and acoustic wavenumbers is similar to the other geometries described previously. This enables us to fix the initial guess in a numerical solution procedure for obtaining the coupled wavenumbers for small and large μ.

5 Conclusions

In this paper, coupled wavenumbers for the systems shown in figure (1) were presented using a unifying asymptotic approach. The detailed derivations of each of the cases were presented in our earlier works. Here, we have discussed certain common features of the formulation and the solution. In each case, we have shown how the coupled equations can be treated as a modification to the uncoupled equations of the structure and of the fluid. The modification depends on a parameter μ termed as the fluid-loading parameter. Physically, μ denotes the mass ratio between the structure and the fluid.

From the coupled equations, we showed that for $\mu = 0$, we get back the *in vacuo* structural wavenumber and the wavenumbers corresponding to the rigid-acoustic duct. Again, for $\mu \to \infty$, we get the wavenumbers of the acoustic duct with pressure-release wall conditions. A perturbation method based on μ (for the case of small μ) or $1/\mu$ (for the case of large μ) gives the coupled wavenumber solutions. Depending on the geometrical complexity, other asymptotic parameters also are used. These are summarized in Table (2).

Problem	Asymptotic parameters
2D waveguide	μ
Circular Cylinder (Axisymmetric mode)	ν & μ
Circular Cylinder (Beam mode)	ϵ & μ
Elliptic cylinder	e and μ

Table 2: Various asymptotic parameters used in this article.

The qualitative nature of the coupled wavenumbers obtained is the same for all the geometric configurations studied and is schematically shown in figure (2). The sign of the correction term in each case indicates the nature of loading of one medium over another. Hence, the following inferences regarding the physical effect of fluid-structure coupling are in order:-

◦ For small μ and for frequencies below the coincidence frequency, the coupled structural wavenumber is greater than the *in vacuo* structural wavenumber. This implies that the effect of the fluid-loading is in the form of additional mass on the structure. For frequencies beyond coincidence, the difference between the coupled and the uncoupled structural wavenumber alternates in sign indicating that the nature of the fluid-loading is alternately mass-like and stiffness-like.

◦ For small μ, there exists a coupled wavenumber branch close to the wavenumber of the acoustic duct with rigid-walls. This coupled rigid-duct wavenumber is lower (greater) than the uncoupled wavenumber for frequencies below (above) the coincidence frequency. Thus, the flexible structural boundary has a mass-effect for frequencies beyond the coincidence frequency and increases the incompressibility of the fluid for frequencies below the coincidence frequency. Also, the cut-on frequencies of the rigid-duct wave modes are increased.

◦ For small μ, at the coincidence frequency, the coupled structural wavenumber joins with the coupled acoustic wavenumber and *vice-versa*. Thus, in contrast to the uncoupled case where the two physical wavenumber branches intersect, in the coupled case there is no intersection but a *gap* is created at the coincidence (also observed by Cabelli [13]).

◦ As μ increases, the coupled wavenumber branches get perturbed further till at large μ, the coupled wavenumber branches are better identified as perturbations to the wavenumbers of the acoustic duct with pressure-release wall conditions. For large μ values, the coupled pressure-release wavenumbers are greater than the uncoupled counterparts below the coincidence frequency and *vice-versa*. Thus, the effect of the flexible structural boundary is opposite to that described previously for small μ.

◦ For large μ, the coupled pressure-release wavenumber equals the uncoupled pressure-release wavenumber at the coincidence frequency. Thus, the flexibility of the structure has virtually no effect at this condition.

Using the asymptotic approach discussed in this article, these common features of the coupled wavenumber characteristics for the various geometries shown in figure (1) are brought out in the schematic of figure (2).

Acknowledgement

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