On a stability of the shear flow with a tangential discontinuity in the presence of a small scatterer

Nikolay Kanev and Mikhail Mironov

Andreev Acoustics Institute, Shvernik, 4, Moscow, 117036 Moscow, Russian Federation
nikolay.kanev@mail.ru

It is well known that a tangential discontinuity of velocity (vortex sheet) in the shear flow is unstable. This instability is convective i.e. all perturbations of the discontinuity grow exponentially but they are carried away by the flow downstream. As a result all the amplitude of sound field generated by the discontinuity will decrease at any fixed point. In the present study we consider the problem when a small scatterer is placed near the discontinuity. It appears this system can be absolutely unstable under the definite position of the scatterer. It means that the appearance of any disturbance of the discontinuity will result in rising of the amplitude of sound field at every point of space. The cause of absolute instability is a feedback provided by the scatterer and the type of instability does not depend on the size of the scatterer. An absolutely unstable system behaves as an autogenerator and radiates tone or multitone sound. So the obtained result explains the possible mechanism of noise generation in systems with tangential discontinuities and gives some hints about noise control in such systems.

1 Introduction

Noise generated by flow with a tangential discontinuity (vortex sheet) is frequently associated with instability of the discontinuity. The radiation properties of the vortex sheet leaving a semi-infinite plate and undergoing a two-dimensional spatial Kelvin-Helmholtz instability is investigated in [1,2]. Note that in the absence of a plate such an instability mode of the vortex sheet generates no sound. The intensity of generated noise is depended only on flow velocity. There are no any certain frequencies characterizing a spectrum of generated noise.

Acoustic tones may be observed in the similar systems with feedback [3,4] when vortices going downstream interact with an obstacle (reflecting boundary, trailing edge of an aperture and so on). As a result of such interaction sound is generated, which propagates upstream and induces new vortices at the point where flow leaves the plate. In the case of instability of flow over apertures (or cavities) it is sufficient to find the conductivity $K_R(\omega)$ of the aperture as a function of the frequency ω of the motion. Instabilities of the aperture flow are determined by poles ω_p of $K_R(\omega)$ in the upper complex frequency plane. If $\operatorname{Im}\omega_p > 0$ the motion of the flow is *absolutely unstable* in that the smallest perturbation of the flow can cause a spontaneous growth of large amplitude motion in the aperture. The real parts of the frequencies ω_p correspond to the Strouhal numbers of self-sustained oscillations. But the size of the vortex sheet is limited by the shape of the aperture. In the case of the semi-infinite vortex sheet the amplitude of the initial perturbation of the flow grows exponentially but it propagates downstream so that after all amplitude of flow oscillation will decrease at any fixed point. In other words the flow is *convectivly unstable* and acoustic tones are not generated.

In this paper we investigate the stability of the laminar flow leaving a semi-infinite rigid plate in the presence of a small scatterer placed outside the flow. We consider a two-dimensional system so that the scatterer is cylindrical. The flow has a tangential discontinuity downstream of the plate edge. The motion of the discontinuity is usually described by means of surface waves traveling on it. A spontaneous disturbance of the discontinuity generates sound waves which are reflected by the scatterer, return to the discontinuity and excite new surface waves. We seek eigenfrequencies of the coupled flow-scatterer system and the conditions determining the type of the instability. The behavior of the system is considered in the linear approximation. If the system is absolutely unstable the amplitude of discontinuity oscillations will grow exponentially in dependence of time. In practice, of course, this exponential growth is curtailed by nonlinear mechanisms ignored by linear perturbation theory.

2 Problem formulation

We consider two-dimensional motion in the (x,z) plane. In the unperturbed state, the fluid in $z>0$ is at rest, while that in $z<0$ streams uniformly with velocity U along the x-axe (Fig. 1). A semi-infinite rigid plate lies in $z=0$, $x<0$. The dipole scatterer is placed in the point (x_0, H) in the resting fluid. The dipole momentum is directed along the z-axe and perpendicular to the tangential discontinuity. The fluid density is ρ and the sound speed is c.

When the scatterer is excited by a force impulse its equation of motion can be written as

$$m\dot{M}(t) = F_0(t) + F(t) \tag{1}$$

where m is the mass of the scatterer, M is its dipole momentum. The momentum of the cylindrical dipole with a radius a is equal to $Va^2/2$, where V is the velocity of the dipole. The scatterer is excited by the force F_0 and fluid acts on the scatterer by the force F. In the simplest case the excited force can be determined as $F_0(t) = f_0\delta(t-t_0)$. Fourier transform of Eq.(1) gives

$$-i\omega m M(\omega) = f_0 e^{i\omega t_0} + F(\omega) \tag{2}$$

The force $F(\omega)$ can be expressed as $F(\omega) = -Z(\omega)v(\omega)$, where $Z(\omega)$ is the impedance.

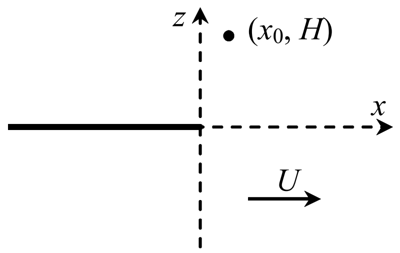

Fig. 1. Problem formulation
Flow is below a semi-infinite rigid plate,
a scatterer is in the resting fluid

So the solution of Eq. (1) in the time domain is

$$M(t) = \frac{1}{2\pi}\int \frac{f_0 e^{-i\omega(t-t_0)}}{-i\omega m + Z(\omega)} d\omega . \quad (3)$$

For $t >> t_0$ the dipole momentum is defined by the singularities of the integrand in Eq.(3). The singularities with $\mathrm{Im}\,\omega > 0$ will dominate as $t \to +\infty$. If they exist the dipole momentum will increase as $M(t) \sim \exp(t\,\mathrm{Im}\,\omega)$ meaning that the system is absolutely unstable. If all singularities are in lower half of the complex frequency plane the system is convective unstable.

The equation for eigenfrequencies of the system in the case of the dipole with $m = 0$ we can write in the simplest form

$$Z(\omega) = 0 . \quad (4)$$

In order to investigate instability of the considered system we have to solve Eq.(4) and find the eigenfrequencies. Their imaginary parts define the type of instability.

3 Impedance of the scatterer

In order to find the impedance of the dipole scatterer placed near the semi-infinite vortex sheet (Fig.1) we consider the small dipole source with unit dipole momentum in the point (x_0, H). The velocity potentials of the resting fluid (denoted by index 1) and the moving fluid (denoted by index 2) satisfy the equations

$$\Delta\varphi_1 + k^2\varphi_1 = 4\pi\frac{\partial}{\partial z}\delta(x - x_0, z - H) . \quad (5)$$

$$\Delta\varphi_2 + k^2\left(1 + \frac{U}{-i\omega}\frac{\partial}{\partial x}\right)^2 \varphi_2 = 0 . \quad (6)$$

where $k = \omega/c$ is the acoustic wavenumber.

The pressure $P_{1,2}$ and the vertical displacement $\eta_{1,2}$ of the fluid are defined by

$$P_1 = i\omega\rho\varphi_1 , \quad (7)$$

$$P_2 = i\omega\rho\left(1 + \frac{U}{-i\omega}\right)\varphi_2 , \quad (8)$$

$$\frac{\partial\varphi_1}{\partial z} = -i\omega\eta_1 , \quad (9)$$

$$\frac{\partial\varphi_2}{\partial z} = -i\omega\left(1 + \frac{U}{-i\omega}\right)\eta_2 . \quad (10)$$

The boundary conditions for $z = 0$ have the following form

$$P_1 = P_2 , \; x > 0 , \quad (11)$$

$$\eta_1 = \eta_2 = 0 , \; x < 0 , \quad (12)$$

$$\eta_1 = \eta_2 = \eta(x) , \; x > 0 , \quad (13)$$

$$\eta(0) = 0 , \; \left.\frac{d\eta}{dx}\right|_{x=0} = 0 . \quad (14)$$

The ordinary technique applied in the similar problems [5] permits to express the solution for the system with a flow based on the solution for the system without a flow. So using the Fourier analysis we can express the pressure in the resting fluid by

$$P_1(x,z) = P_d + P_r + P_t , \quad (15)$$

where the pressure P_0 is the direct field of the scatterer, the pressure P_r is the field reflected from the absolutely rigid boundary $z = 0$, and the pressure P_t is the field generated by the motion of the discontinuity. The last one is given by

$$P_t = -2i\rho\omega^2 \int \frac{\kappa_2}{\kappa_1}\frac{g_0(\xi)}{\kappa_1(1 - U\xi/\omega)^2 + \kappa_2} e^{i\xi x + i\kappa_1 z} d\xi , \quad (16)$$

where $\kappa_1 = \sqrt{k^2 - \xi^2}$, $\kappa_2 = \sqrt{(k - U\xi/c)^2 - \xi^2}$.

In two last expressions we have to choose the value with $\mathrm{Im}\,\kappa_1 > 0$ и $\mathrm{Im}\,\kappa_2 > 0$. The function $g_0(\xi)$ is the Fourier component of the displacement $\eta(x)$ in the absence of the flow, i.e. under condition $U = 0$, and is given by

$$g_0(\xi) = \frac{1}{2\pi}\int \eta_0(x) e^{i\xi x} dx , \quad (17)$$

where $\eta_0(x) = \eta(x)|_{U=0}$.

The solution of the diffraction problem can be found by means of Wiener-Hopf analysis [6]. Without flow the velocity potential in $z > 0$ is equal to

$$\varphi_1 = -i\pi k H_1^{(1)}(kR)\sin\theta + $$
$$+ \frac{1}{2\pi}\int_{i\tau-\infty}^{i\tau+\infty} i\beta d\beta \int_{i\psi-\infty}^{i\psi+\infty} d\xi \frac{e^{-i\xi x + i\kappa_1 z} e^{-i\beta x_0 + i\mu H}}{\sqrt{\xi - k}\sqrt{\beta - k}(\xi + \beta)} , \quad (18)$$

where $-\mathrm{Im}\,k < \psi < -\tau < \mathrm{Im}\,k$, $\mu = \sqrt{\beta^2 - k^2}$, $\mathrm{Im}\,\mu > 0$, $R^2 = (x - x_0)^2 + (z - H)^2$, $\tan\theta = (z - H)/(x - x_0)$.

In order to find the impedance of the scatterer $Z(\omega)$ we have to consider the dipole with a finite size. Let us assume that the dipole scattetrer is a cylinder with a radius a so that $a << H$ and $ka << 1$. This situation corresponds to incompressible flow which will be considered below.

The impedance of the dipole placed near absolutely rigid boundary at the distance H ($kH << 1$) is expressed by

$$Z_0 = \frac{F_d + F_r}{M} = -2i\omega\rho\pi\left(1 - \frac{a^2}{H^2}\right) . \quad (19)$$

The active part of the impedance Z_0 is much smaller than the reactive part and can be neglected. Moreover the last term in Eq.(19) is small as well, so we can take $Z_0 = -2i\omega\rho\pi$.

The pressure P_t defines the force F_t acting on the scatterer in the following way

$$F_t(\omega) = -\pi a^2 \left.\frac{\partial P_t}{\partial z}\right|_{(x_0,H)} \approx$$

$$\approx -2\pi a^2\rho\omega^2 \int \kappa_2 \frac{g_0(\xi)}{\kappa_1(1 - U\xi/\omega)^2 + \kappa_2} e^{i\xi x_0 + i\kappa_1 H} d\xi . \quad (20)$$

In the case of small distance between the source and the plane edge in comparison with wavelength an approximation of incompressible fluid is possible. At this approximation $k \to 0$ and $\kappa_1 = \kappa_2 = \kappa = \sqrt{-\xi^2}$. So the dipole impedance provided by the flow is given by

$$Z_t = \frac{F_t}{M} = -2\pi a^2 \rho \omega^2 \int \frac{g_0(\xi)}{(1 - U\xi/\omega)^2 + 1} e^{i\xi x_0 + i\kappa H} d\xi . \quad (21)$$

From Eq.(18) we can find the Fourier component of the vertical displacement in $z = 0$ under condition $k \to 0$

$$g_0(\xi) = -\frac{\kappa}{\omega}\left(e^{-i\xi x_0 + i\kappa_1 H} - \right. \\ \left. - \frac{i}{2\pi\sqrt{-\xi}} \int \frac{\sqrt{\beta}}{\beta - \xi} e^{-i\beta x_0 + i\mu H} d\beta \right). \quad (22)$$

Substituting Eq.(22) in Eq.(21) we obtain the dipole impedance provided by the flow Z_t.

The integrand in Eq.(21) has two poles $\xi_{1,2} = \frac{\omega}{U}(1 \pm i)$. In accordance with causality principle the path of integration in Eq.(21) must pass below these poles. The integration contour may be displaced upwards towards the real axis and Eq.(21) may be written in the following form

$$Z_t = -2\pi a^2 \rho \omega^2 \times \\ \times \left(\frac{\pi\omega}{U} \sum_{j=1,2} g_0(\xi_j) \gamma(\xi_j) e^{i\xi_j x_0 - i\kappa(\xi_j) H} + I_R \right), \quad (23)$$

where $\gamma(\xi_j) = 1$ if $\mathrm{Im}(\xi_j) < 0$ and $\gamma(\xi_j) = 0$ if $\mathrm{Im}(\xi_j) > 0$, I_R is the result of integration along the real axis.

The impedances Z_0 and Z_t define the total impedance of the source $Z = Z_0 + Z_t$.

4 Eigenfrequencies of the system

The equation for eigenfrequencies of the considered system has the simple form $Z = 0$. Substituting Eq.(19) and Eq.(23) in this equation we have

$$1 - i\omega a^2 \left(\frac{\pi\omega}{U} \sum_{j=1,2} g_0(\xi_j) \gamma(\xi_j) e^{i\xi_j x_0 - i\kappa(\xi_j) H} + I_R \right) = 0 . \quad (24)$$

Eq.(24) is suitable for calculation of eigenfrequencies. Nevertheless this equation could be simplified. First of all the numerical estimation shows that the absolute value of $\omega a^2 I_R$ is much less than 1. So we can neglect by I_R. Since our interest is to determine the condition of absolute instability it is important to find the eigenfrequencies in the upper half of the ω plane. For $\mathrm{Im}\,\omega > 0$ only one of two poles $\xi_{1,2}$ is in the lower half of the ξ plane. Hence Eq.(24) transforms in

$$1 - i\pi a^2 \frac{\omega^2}{U} g_0(\xi_2) e^{i\xi_2 x_0 - i\kappa(\xi_2) H} = 0 . \quad (25)$$

Another simplification is possible in Eq. (22). The first term in Eq.(22) may not be taken into account because it corresponds to the flow without semi-infinite rigid plate. Obviously such a system is convective unstable and its eigenfrequencies are under the real axis of the ω plane. As we seek the absolute instability we can substitute only second term of Eq.(22) in Eq. (25). Putting the dimensionless frequency $p = \omega H/U$, distance $s = x_0/H$ and wavenumber $b = \beta H$ we obtain the final equation for the eigenfrequencies of the considered system

$$1 - \frac{\sqrt{1-i}}{2}\left(\frac{a}{H}\right)^2 p\sqrt{p} e^{p(1+i)s - p(1-i)} \int \frac{\sqrt{b}}{b - p(1-i)} e^{-ib\sigma - |b|} db = 0 . \quad (26)$$

where the integration is along the real axis.

The roots of Eq. (26) have been calculated for the scatterer size $a/H = 0.1$. The evolution of three eigenfrequencies under increasing distance s is shown on Fig. 2. The value of s is changing from 0 to 2. We see that at $s = 0$ all eigenfrequencies are below the real axis. Under increasing s (that corresponds to the motion the scatterer downstream) they approach the real axis and under certain value of s, which is close to 1, cross the real axis. That means the system becomes absolutely unstable.

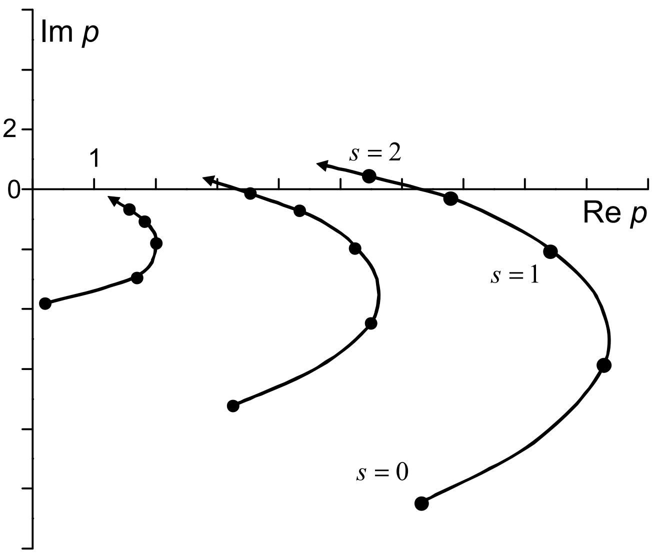

Fig. 2. Eigenfrequencies of the system in dependence of parameter s

5 Conclusion

The small scatterer can change the nature of the unstable motion of flow leaving the rigid plate. The presence of any scatterer in the area marked on Fig. 3 makes flow absolutely unstable. The obtained result does not depend on the scatterer size. From physical point of view it is obvious that this result is similar for different types of scatteters (monopole, dipole and so on). The role of a scatterer is to provide a strong enough feedback to excite surface waves traveling on the discontinuity of flow (vortex sheet).

The obtained result permits to make a conclusion that in systems with the fixed beginning of a vortex sheet there is an "area of absolute instability". In order to prevent self-sustained oscillations of flow and tone noise generation it is necessary to avoid backward scattering from this zone.

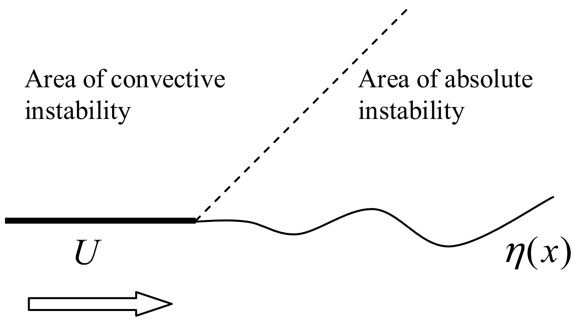

Fig. 3. Areas of convective and absolute instability

Acknowledgments

This work has been supported by Russian Foundation for Basic Research under Grant No. 08-02-00505.

References

[1] D.G. Crighton, "Radiation properties of the semi-infinite vortex sheet", *Proc. Roy. Soc.* A330, 185-198 (1972)

[2] S.A. Orszag, S.C. Crow, "Instability of a vortex sheet leaving a semi-infinite plate", *Stud. Appl. Math.* 49, 167-181 (1970)

[3] M.S. Howe, "Low Strouhal number instabilities of flow over apertures and wall cavities", *J. Acoust. Soc. Am.* 102, 772-780 (1997)

[4] S.M. Grace, "Prediction of low frequency tones produces by flow through a duct with a gap", *J. Sound and Vib.* 229, 859-878 (2000)

[5] A.D. Lapin, M.A. Mironov, "Admittance of a groove on a rigid surface under grazing flow", *Phys. Acoust.* 49, 75-80 (2003)

[6] B. Noble, "Methods based on the Wiener-Hpf technique", *London: Pergamon* (1958)