On a stability of the shear flow with a tangential discontinuity in the presence of a small scatterer

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It is well known that a tangential discontinuity of velocity (vortex sheet) in the shear flow is unstable. This instability is convective i.e. all perturbations of the discontinuity grow exponentially but they are carried away by the flow downstream. As a result all the amplitude of sound field generated by the discontinuity will decrease at any fixed point. In the present study we consider the problem when a small scatterer is placed near the discontinuity. It appears this system can be absolutely unstable under the definite position of the scatterer. It means that the appearance of any disturbance of the discontinuity will result in rising of the amplitude of sound field at every point of space. The cause of absolute instability is a feedback provided by the scatterer and the type of instability does not depend on the size of the scatterer. An absolutely unstable system behaves as an autogenerator and radiates tone or multitone sound. So the obtained result explains the possible mechanism of noise generation in systems with tangential discontinuities and gives some hints about noise control in such systems.

1 Introduction

Noise generated by flow with a tangential discontinuity (vortex sheet) is frequently associated with instability of the discontinuity. The radiation properties of the vortex sheet leaving a semi-infinite plate and undergoing a two-dimensional spatial Kelvin-Helmholtz instability is investigated in [1,2]. Note that in the absence of a plate such an instability mode of the vortex sheet generates no sound. The intensity of generated noise is depended only on flow velocity. There are no any certain frequencies characterizing a spectrum of generated noise.

Acoustic tones may be observed in the similar systems with feedback [3,4] when vortices going downstream interact with an obstacle (reflecting boundary, trailing edge of an aperture and so on). As a result of such interaction sound is generated, which propagates upstream and induces new vortices at the point where flow leaves the plate. In the case of instability of flow over apertures (or cavities) it is sufficient to find the conductivity $K_\omega(\omega)$ of the aperture as a function of the frequency $\omega$ of the motion. Instabilities of the aperture flow are determined by poles $\omega_p$ of $K_\omega(\omega)$ in the upper complex frequency plane. If $\text{Im}\omega_p > 0$ the motion of the flow is absolutely unstable in that the smallest perturbation of the flow can cause a spontaneous growth of large amplitude motion in the aperture. The real parts of the frequencies $\omega_p$ correspond to the Strouhal numbers of self-sustained oscillations. But the size of the vortex sheet is limited by the shape of the aperture. In the case of the semi-infinite vortex sheet the amplitude of the initial perturbation of the flow grows exponentially but it propagates downstream so that after all amplitude of flow oscillation will decrease at any fixed point. In other words the flow is convectively unstable and acoustic tones are not generated.

In this paper we investigate the stability of the laminar flow leaving a semi-infinite rigid plate in the presence of a small scatterer placed outside the flow. We consider a two-dimensional system so that the scatterer is cylindrical. The flow has a tangential discontinuity downstream of the plate edge. The motion of the discontinuity is usually described by means of surface waves traveling on it. A spontaneous disturbance of the discontinuity generates sound waves which are reflected by the scatterer, return to the discontinuity and excite new surface waves. We seek eigenfrequencies of the coupled flow-scatterer system and the conditions determining the type of the instability. The behavior of the system is considered in the linear approximation. If the system is absolutely unstable the amplitude of discontinuity oscillations will grow exponentially in dependence of time. In practice, of course, this exponential growth is curtailed by nonlinear mechanisms ignored by linear perturbation theory.

2 Problem formulation

We consider two-dimensional motion in the $(x,z)$ plane. In the unperturbed state, the fluid in $z > 0$ is at rest, while that in $z < 0$ streams uniformly with velocity $U$ along the $x$-axe (Fig. 1). A semi-infinite rigid plate lies in $z = 0$, $x < 0$. The dipole scatterer is placed in the point $(x_0,H)$ in the resting fluid. The dipole momentum is directed along the $z$-axe and perpendicular to the tangential discontinuity. The fluid density is $\rho$ and the sound speed is $c$.

When the scatterer is excited by a force impulse its equation of motion can be written as

$$mM(t) = F_0(t) + F(t)$$

(1)

where $m$ is the mass of the scatterer, $M$ is its dipole momentum. The momentum of the cylindrical dipole with a radius $a$ is equal to $\frac{1}{2}Va^2/2$, where $V$ is the velocity of the dipole. The scatterer is excited by the force $F_0$ and fluid acts on the scatterer by the force $F$. In the simplest case the excited force can be determined as $F_0(t) = f_0\delta(t - \tau_0)$. Fourier transform of Eq.(1) gives

$$-im\omega M(\omega) = f_0e^{i\omega\tau_0} + F(\omega)$$

(2)

The force $F(\omega)$ can be expressed as $F(\omega) = -Z(\omega)v(\omega)$, where $Z(\omega)$ is the impedance.
So the solution of Eq. (1) in the time domain is
\[ M(t) = \frac{1}{2\pi} \int f_0 e^{-i\omega t} d\omega. \] (3)

For \( t \gg t_0 \) the dipole momentum is defined by the singularities of the integrand in Eq.(3). The singularities with \( \Im \omega > 0 \) will dominate as \( t \to +\infty \). If they exist the dipole momentum will increase as \( M(t) \sim \exp(t \Im \omega) \) meaning that the system is absolutely unstable. If all singularities are in lower half of the complex frequency plane the system is convectively unstable.

The equation for eigenfrequencies of the system in the case of the dipole with \( m = 0 \) can we write in the simplest form
\[ Z(\omega) = 0. \] (4)

In order to investigate instability of the considered system we have to solve Eq.(4) and find the eigenfrequencies. Their imaginary parts define the type of instability.

3 Impedance of the scatterer

In order to find the impedance of the scatterer placed near the semi-infinite vortex sheet (Fig.1) we consider the small dipole source with unit dipole momentum in the point \((x_0,H)\). The velocity potentials of the resting fluid (denoted by index 1) and the moving fluid (denoted by index 2) satisfy the equations
\[ \Delta \phi_1 + k^2 \phi_1 = 4\pi \frac{\partial}{\partial z} \delta(x-x_0,z-H). \] (5)
\[ \Delta \phi_2 + k^2 \left[ 1 + \frac{U}{-i\omega} \right] \phi_2 = 0. \] (6)

where \( k = \omega/c \) is the acoustic wavenumber.

The pressure \( P_{1,2} \) and the vertical displacement \( \eta_{1,2} \) of the fluid are defined by
\[ P_1 = i\omega \phi_1, \] (7)
\[ P_2 = i\omega \left( 1 + \frac{U}{-i\omega} \right) \phi_2, \] (8)
\[ \frac{\partial \phi_1}{\partial z} = -i\omega \eta_1, \] (9)
\[ \frac{\partial \phi_2}{\partial z} = -i\omega \left( 1 + \frac{U}{-i\omega} \right) \eta_2. \] (10)

The boundary conditions for \( z = 0 \) have the following form
\[ P_1 = P_2, \quad x > 0, \] (11)
\[ \eta_1 = \eta_2 = 0, \quad x < 0, \] (12)
\[ \eta_1 = \eta_2 = \eta(x), \quad x > 0, \] (13)
\[ \eta(0) = 0, \quad \frac{d\eta}{dx} \bigg|_{x=0} = 0. \] (14)

The ordinary technique applied in the similar problems [5] permits to express the solution for the system with a flow based on the solution for the system without a flow. So using the Fourier analysis we can express the pressure in the resting fluid by
\[ P_1(x,z) = P_0 + P_1 + P_1', \] (15)
where the pressure \( P_0 \) is the direct field of the scatterer, the pressure \( P_1 \) is the field reflected from the absolutely rigid boundary \( z = 0 \), and the pressure \( P_1' \) is the field generated by the motion of the discontinuity. The last one is given by
\[ P_1' = -2i\rho_0 \xi \left[ \kappa_2 \frac{g_0(\xi)}{\kappa_1(1-U \xi /\omega)^2} + \kappa_2 \xi \right] e^{i\phi_{1} + i\kappa_1 \beta} d\xi, \] (16)

where \( \kappa_1 = \sqrt{k^2 - \xi^2}, \quad \kappa_2 = \sqrt{(k-U \xi /c)^2 - \xi^2}. \)

In two last expressions we have to choose the value with \( \Im \kappa_1 > 0 \) and \( \Im \kappa_2 > 0 \). The function \( g_0(\xi) \) is the Fourier component of the displacement \( \eta(x) \) in the absence of the flow, i.e. under condition \( U = 0 \), and is given by
\[ g_0(\xi) = \frac{1}{2\pi} \int \eta_0(x)e^{i\xi x} dx, \] (17)
where \( \eta_0(x) = \eta(x)|_{U=0}. \)

The solution of the diffraction problem can be found by means of Wiener-Hopf analysis [6]. Without flow the velocity potential in \( z > 0 \) is equal to
\[ \varphi_1 = -i \mu H \left( i k \right) \sin \theta + \frac{1}{2\pi} \int \frac{f d\beta}{\sqrt{k^2 - \xi^2}} \frac{\xi - e^{i\xi \beta} - \xi^2 e^{i\xi \beta}}{\sqrt{k^2 - \xi^2 - k \beta}}, \] (18)

where \(-\Im k < \psi < -\pi < \Im k, \mu = \sqrt{\beta^2 - k^2}, \Im \mu > 0, \quad R^2 = (x-x_0)^2 + (z-H)^2, \tan \theta = (z-H)/(x-x_0)\).

In order to find the impedance of the scatterer \( Z(\omega) \) we have to consider the dipole with a finite size. Let us assume that the dipole scatterer is a cylinder with a radius \( a \) so that \( a << H \) and \( ka << 1 \). This situation corresponds to incompressible flow which will be considered below.

The impedance of the dipole placed near absolutely rigid boundary at the distance \( H \) \( (kh << 1) \) is expressed by
\[ Z_0 = \frac{F_0}{M} = -2i\omega \pi \left( 1 - \frac{a^2}{H^2} \right). \] (19)

The active part of the impedance \( Z_0 \) is much smaller than the reactive part and can be neglected. Moreover the last term in Eq.(19) is small as well, so we can take \( Z_0 = -2i\omega \pi \).

The pressure \( P_1 \) defines the force \( F_1 \) acting on the scatterer in the following way
\[ F_1(\omega) = -ma^2 \frac{\partial P_1}{\partial z} \bigg|_{x=0} = -2ma^2 \rho_0 \xi \left[ \kappa_1 \frac{g_0(\xi)}{\kappa_1(1-U \xi /\omega)^2} + \kappa_2 \right] e^{i\phi_{1} + i\kappa_1 \beta} d\xi. \] (20)

In the case of small distance between the source and the plane edge in comparison with wavelength an approximation of incompressible fluid is possible. At this
approximation $k \to 0$ and $\kappa_1 = \kappa_2 = \kappa = \sqrt{-\xi^2}$. So the dipole impedance provided by the flow is given by

$$Z_i = \frac{F_i}{M} = -2\pi i \rho \omega^2 \int \frac{g_0(\xi)}{1 - U/\omega} e^{i\phi_0 + i\omega t} d\xi. \quad (21)$$

From Eq.(18) we can find the Fourier component of the vertical displacement in $z = 0$ under condition $k \to 0$

$$g_0(\xi) = -\frac{\omega}{\rho \omega^2} e^{-i\phi_0 + i\omega t} - \frac{i}{2\pi \sqrt{\xi}} \left( \frac{\beta}{\beta - \xi} e^{-i\phi_0 + i\omega t} d\beta \right). \quad (22)$$

Substituting Eq.(22) in Eq.(21) we obtain the dipole impedance provided by the flow $Z_i$.

The integrand in Eq.(21) has two poles $\xi_{1,2} = \omega/\beta(1 \pm i)$. In accordance with causality principle the path of integration in Eq.(21) must pass below these poles. The integration contour may be displaced upwards towards the real axis and in Eq.(21) may be written in the following form

$$Z_i = -\pi i \rho \omega^2 \times \left( \frac{\pi \omega}{U} \sum_{j=1,2} g_0(\xi_j) \gamma(\xi_j) e^{i(\xi_j \gamma_j - \xi_j \gamma_j) + I_g} \right), \quad (23)$$

where $\gamma(\xi_j) = 1$ if $\text{Im}(\xi_j) < 0$ and $\gamma(\xi_j) = 0$ if $\text{Im}(\xi_j) > 0$. $I_g$ is the result of integration along the real axis.

The impedances $Z_0$ and $Z_i$ define the total impedance of the source $Z = Z_0 + Z_i$.

4 Eigenfrequencies of the system

The equation for eigenfrequencies of the considered system has the simple form $Z = 0$. Substituting Eq.(19) and Eq.(23) in this equation we have

$$1 - i \omega \beta \left( \frac{\pi \omega}{U} \sum_{j=1,2} g_0(\xi_j) \gamma(\xi_j) e^{i(\xi_j \gamma_j - \xi_j \gamma_j) + I_g} \right) = 0. \quad (24)$$

Eq.(24) is suitable for calculation of eigenfrequencies. Nevertheless this equation could be simplified. First of all the numerical estimation shows that the absolute value of $\omega \beta I_g$ is much less than 1. So we can neglect by $I_g$. Since our interest is to determine the condition of absolute instability it is important to find the eigenfrequencies in the upper half of the $\omega$ plane. For $Im \omega > 0$ only one of two poles $\xi_{1,2}$ is in the lower half of the $\xi$ plane. Hence Eq.(24) transforms in

$$1 - i \omega \beta \frac{\pi \omega}{U} g_0(\xi_2) e^{i(\xi_2 \gamma_2 - \xi_2 \gamma_2)} = 0. \quad (25)$$

Another simplification is possible in Eq. (22). The first term in Eq.(22) may not be taken into account because it corresponds to the flow without semi-infinite rigid plate. Obviously such a system is convective unstable and its eigenfrequencies are under the real axis of the $\omega$ plane. As we seek the absolute instability we can substitute only second term of Eq.(22) in Eq. (25). Putting the dimensionless frequency $p = \omega H/U$, distance $s = x_0/H$ and wavenumber $b = \beta H$ we obtain the final equation for the eigenfrequencies of the considered system

$$1 - \sqrt{\frac{i}{2}} \left( \frac{a}{H} \right)^2 p \sqrt{p e^{p(1+i)} - 1} \int \frac{\sqrt{b}}{b - p(1 - i)} e^{-\omega s} p I_{db} = 0. \quad (26)$$

where the integration is along the real axis.

The roots of Eq. (26) have been calculated for the scatterer size $a/H = 0.1$. The evolution of three eigenfrequencies under increasing distance $s$ is shown on Fig. 2. The value of $s$ is changing from 0 to 2. We see that at $s = 0$ all eigenfrequencies are below the real axis. Under increasing $s$ (that corresponds to the motion along the scatterer downstream) they approach the real axis and under certain value of $s$, which is close to 1, cross the real axis. That means the system becomes absolutely unstable.

![Fig. 2. Eigenfrequencies of the system in dependence of parameter $s$](image-url)
Fig. 3. Areas of convective and absolute instability

Acknowledgments

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References