Acoustics'08 Paris
June 29-July 4, 2008
www.acoustics08-paris.org

Analytical validation of time-step interpolation in transient insular nodal analysis

Tom De Rybel, José Marti and Murray Hodgson

The University of British Columbia, Department of Electrical and Computer Engineering, 2332 Main Mall, Vancouver, BC, Canada V6T 1Z4
tomr@ece.ubc.ca

The transient insular nodal analysis method (TINA) combines elements from the finite-differences (FD) and transmission-line matrix methods (TLM) in one unified approach. As with most time-domain finite-difference (TD-FD) methods, TINA uses time-decoupled cells, avoiding the need for solving large system matrices. The time-decoupled cells allow for easy parallelisation, and the solution of large systems in detail. Due to the use of an exact transmission-line model in the cells, wave propagation can be computed without the need for discretisation of the equations, nor the use of prediction, yielding an unconditionally stable method. Boundary conditions are implicit, and are solely defined by the wave speed and characteristic impedance of the medium. One key difference is how cells which transmission time is not an integer multiple of the simulation time step are incorporated in the TINA simulation. These mismatches occur due to the varying wave speeds of the different media in the cells. In TINA, the match is obtained through interpolation, as opposed to the stub-matching methodology employed in TLM. In this paper, we will demonstrate the validity of the interpolation approach analytically, as well as compare the interpolation method to a theoretical case.

1 Introduction

Matrix methods, such as the transmission-line matrix method (TLM) pioneered by Johns [1], are commonly used in many disciplines. Since their inception, one of the main issues bas been to maintain synchronicity of the numerical simulation when different materials are present and the propagation times are not integer multiples of each other [2, 3]. In TLM, stubs are used to reduce the wave speed in some of the materials and allow for a common time step. Changes in the discretisation mesh itself are also used in some cases. Similar methods are also proposed in time-domain finite-difference (TD-FD) methods such as the classic Yee method [4] or the different formulation of this approach proposed by Chen [5]. The latter, together with concepts from the electro-magnetic transient program (EMTP) [6], is the base concept used in the transient insular nodal analysis (TINA) simulator presented here [7].

Although stubs are computationally efficient, their use introduces undesired effects [3]. First, the bandwidth of the material is reduced. Second, the response of the stub-matched line exhibits a parasitic transient behaviour due to the reflections on the stub.

The second of these disadvantages can be overcome by a different approach to synchronism through the use of interpolation. This method is commonly used in EMTP-style simulations of electrical power systems [6].

Such an interpolated transmission-line model can be readily used in a TD-FD scheme. In TINA, no modifications are required as the solutions are obtained though nodal analysis, which is the same method used in EMTP simulations [8]. This allows for a direct implementation of the interpolated line-model approach and through this, a simplified and effective approach to system-wide synchronisation without the use of stubs or an irregular mesh.

In this paper, the interpolated transmission line will be derived from the ideal model in the time domain. Its potential accuracy will be shown analytically for the 1D case in the frequency domain by comparison with an exact line model. The latter is normally used in both TLM and TD-FD solutions when no matching is required. Time-domain comparisons will also be made.

2 The Bergeron Line Model

Both the TLM and TD-FD families of simulators are based on the underlying concept of wave propagation, be it conceptually through a scattering (Huygens' principle) approach [1], or a direct discretised solution of Maxwell's equations [4, 5].

When solved for an 1D case in a lossless, non-dispersive, homogeneous medium, the solution for a travelling wave can be obtained from Maxwell's equations [9]:

$$\frac{\partial^2 E_x(z,t)}{\partial z^2} = \epsilon\mu \frac{\partial^2 E_x(z,t)}{\partial t^2} \quad v = \frac{1}{\sqrt{\mu\epsilon}} \quad Z_c = \sqrt{\frac{\mu}{\epsilon}} \tag{1}$$

This wave equation shares the same form as the 1D acoustic plane-wave equation [10, 11].

When solved for sinusoidal waves (d'Allembert solution), and combining the current/voltage equation pairs for forward and backward waves [9, 8, 7], the following time-domain equations are obtained for the model in Fig.1, with τ the propagation time of the line and Z_c the characteristic impedance:

$$e_{kh}(t) = v_m(t-\tau) + Z_c i_m(t-\tau) \tag{2a}$$
$$e_{mh}(t) = v_k(t-\tau) + Z_c i_k(t-\tau) \tag{2b}$$

$$v_m(t) - Z_c i_m(t) = e_{mh}(t) \tag{3a}$$
$$v_k(t) - Z_c i_k(t) = e_{kh}(t) \tag{3b}$$

Note the sign change on the current in the second pair of equations. This is due to the fact that the model described in Fig.1 has the current set to flow into the model at both ends. As such, a current that enters the model at node K inverts sign when it exits at node M (counter to the indicated current). The sign change corrects for this and simplifies the use of the model by keeping the current flow consistent, regardless of which side of the model is used for injection and reception of the forward and backward waves.

This model, known as the Bergeron model [12], is an exact solution to the wave equation in the 1D lossless case, and can be directly implemented in a digital computer [8]. An additional advantage of the model is that both terminals k and m are galvanically decoupled in time, allowing a local solution. TLM and TD-FD methods use, at least implicitly, this model for the ideal, lossless case.

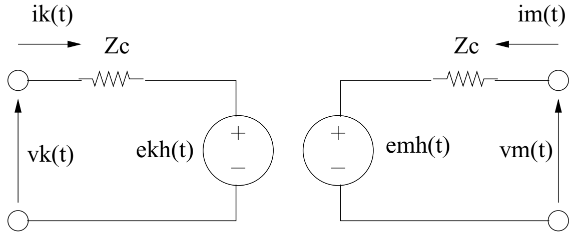


Figure 1: Lossless Line Model

3 The Interpolated Line Model

In a discrete calculation, one has to choose a time step for the simulation. As long as this time step is equal to, or an integer multiple of, the transmission time of the line, the solution can be found. However, in a practical system, multiple materials are frequently required for a sufficiently accurate representation of reality. These materials rarely have wave speeds that work out to integer multiples of each other's propagation time. As such, there is a problem of synchronicity, where different time steps would be required within the simulation [2, 3].

In EMTP simulations of power systems, interpolation is frequently used to overcome the problem of synchronicity of the transmission lines [6, 13].

When implemented on a digital computer, the expressions of Eqs (2a,2b) are programmed so that the past values in the equations, referred to by the indices $(t-\tau)$, are available in program memory in order for the results at the current time step (t) to be computed. For a given τ, which is defined by the length of the line l and its wave speed v, we can find the number of history values that need to be stored in memory from the simulation time step Δt:

$$\tau = \frac{v}{l} \quad (4)$$

$$\textit{history depth} = \frac{\tau}{\Delta t} \quad (5)$$

When the τ of the line is an integer multiple of the simulation time step, this history depth will be an integer number, and the value is available in memory since data is saved in the history storage at each time step.

However, when the τ of the line is no longer an integer multiple of the time step, we require access to history values that lie in-between exiting data in order to compute the next contribution at the current time step (t). The history memory depth is then rounded-up to the next integer value to accommodate the time-step bound data and interpolation is used to obtain the required history value.

In Fig.2, a simulation time step of $\Delta t = 0.02$ ms and a travelling time of $\tau = 0.10$ ms are used. It can be seen that, for a simulation time (t) of 0.18 ms, the required history value of 0.08 ms is available in the table and the current value can be readily computed.

In Fig.3, a simulation time step of $\Delta t = 0.03$ ms is used. This time, the required history value of 0.08 ms is not available in the table, as it lies in between the available history values of 0.06 ms and 0.09 ms.

When the change between two such history values is relatively small, it can be seen that the use of interpolation in time between the history values may be of use to find a good approximation for the variable at the required time. In EMTP, and TINA, linear interpolation is used for computational efficiency.

t(ms)	Vk	Ik	Vm	Im
0	~	~	~	~
0.02	~	~	~	~
0.04	~	~	~	~
0.06	~	~	~	~
0.08	~	~	~	~
0.10	~	~	~	~
0.12	~	~	~	~
0.14	~	~	~	~
0.16	~	~	~	~
0.18	~	~	~	~

Figure 2: History Values in Memory, $\Delta t = 0.02$ ms

t(ms)	Vk	Ik	Vm	Im
0	~	~	~	~
0.03	~	~	~	~
0.06	~	~	~	~
0.09	~	~	~	~
0.12	~	~	~	~
0.15	~	~	~	~
0.18	~	~	~	~

Figure 3: History Values in Memory, $\Delta t = 0.03$ ms

Applying linear interpolation [14] to the line model from Eqs (2a,2b), the interpolated equation pair becomes:

$$\begin{aligned} e_{kh}(t) &= v_m(t-\tau_{int}) \\ &\quad + r\left[v_m(t-\tau_{int}-\Delta t) - v_m(t-\tau_{int})\right] \\ &\quad + Z_c i_m \\ &\quad + Z_c r\left[i_m(t-\tau_{int}-\Delta t) - i_m(t-\tau_{int})\right] \end{aligned} \quad (6a)$$

$$\begin{aligned} e_{mh}(t) &= v_k(t-\tau_{int}) \\ &\quad + r\left[v_k(t-\tau_{int}-\Delta t) - v_k(t-\tau_{int})\right] \\ &\quad + Z_c i_k \\ &\quad + Z_c r\left[i_k(t-\tau_{int}-\Delta t) - i_k(t-\tau_{int})\right] \end{aligned} \quad (6b)$$

The known history values are stored at $(t - \tau_{int})$ and $(t - \tau_{int} - \Delta t)$, where τ_{int} is given by Eq (7) and is the line travelling time expressed as an integer multiple of the simulation time step. % is the modulo division.

$$\tau_{int} = \tau - (\tau\%\Delta t) \quad (7)$$

r is the distance from the $(t - \tau_{int})$ known value within the interpolated interval so that $0 \leq r < 1$, and is given by:

$$r = \frac{\tau\%\Delta t}{\Delta t} \quad (8)$$

4 Frequency Domain Error Analysis

Since the errors associated with the interpolation are due to a low-pass filter effect, the analysis was performed in the frequency domain. Substituting the history sources for the terminal conditions and writing both the ideal-line and interpolated-line equation pairs in phasor form:

$$\overline{V}_k - Z_c\overline{I}_k = \left(\overline{V}_m + Z_c\overline{I}_m\right) e^{-j\omega\tau} \tag{9a}$$
$$\overline{V}_m - Z_c\overline{I}_m = \left(\overline{V}_k + Z_c\overline{I}_k\right) e^{-j\omega\tau} \tag{9b}$$

$$\overline{V}_k - Z_c\overline{I}_k = \left(\overline{V}_m + Z_c\overline{I}_m\right) e^{-j\omega\tau_{int}} S \tag{10a}$$
$$\overline{V}_m - Z_c\overline{I}_m = \left(\overline{V}_k + Z_c\overline{I}_k\right) e^{-j\omega\tau_{int}} S \tag{10b}$$

$$S = \left[1 + r\left(e^{-j\omega\Delta t} - 1\right)\right] \tag{11}$$

Eqs (10a,10b), representing the interpolated line, have two parts: an ideal line with a transmission time set to an integer multiple of the time step, and the travelling-time correction performed by multiplying with a time-shifting function S. It is this function that will change the frequency response of the model. r is the interpolation factor defined in Eq (8). When r is zero (i.e., no interpolation), S becomes unity and the response is equal to the ideal line-model with length τ. If r were to equal one, the response would be equal to a longer ideal line with τ one Δt larger then before. Again, no interpolation would be used. For all other values of r, interpolation (thus, time-shifting) occurs.

In practice, r cannot equal one, as the integer division operator in Eq (8) would wrap around and make it zero. The line length would then be adjusted to an integer multiple, and thus not require interpolation.

In order to find the frequency and phase responses of the interpolated model, Eqs (9a,9b,10a,10b) were solved using Maxima[1] for both open-circuit and short-circuit boundary conditions on node M [15]. Transfer functions can now be obtained:

- Open ideal line $\rightarrow \overline{I}_m = 0$

$$\frac{\overline{V}_m}{\overline{V}_k} = \frac{2e^{j\omega\tau}}{e^{2j\omega\tau} + 1} \tag{12}$$

- Shorted ideal line $\rightarrow \overline{V}_m = 0$

$$\frac{\overline{I}_m}{\overline{I}_k} = -\frac{e^{2j\omega\tau} + 1}{2e^{j\omega\tau}} \tag{13}$$

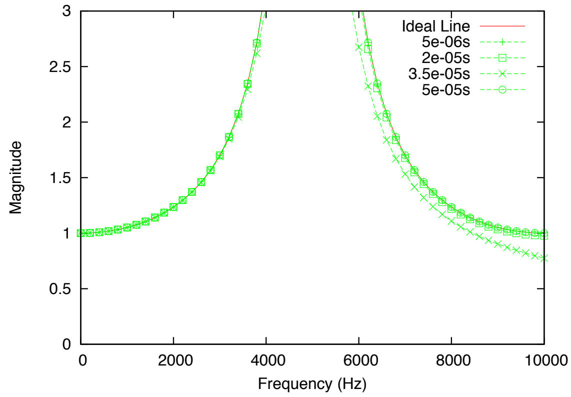
- Open interpolated line $\rightarrow \overline{I}_m = 0$

$$\frac{\overline{V}_m}{\overline{V}_k} = -\frac{N}{D} \tag{14}$$

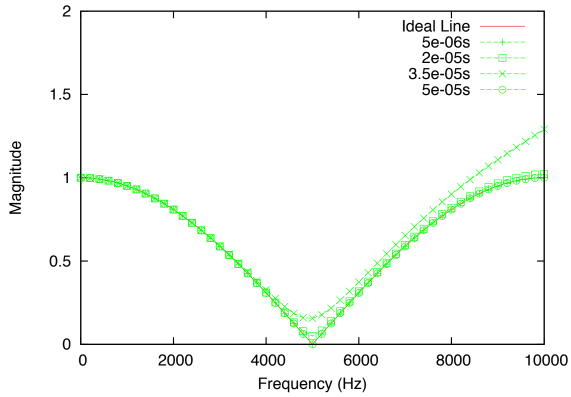
- Shorted interpolated line $\rightarrow \overline{V}_m = 0$

$$\frac{\overline{I}_m}{\overline{I}_k} = \frac{D}{N} \tag{15}$$

[1] An open-source computer algebra system (`maxima.sourceforge.net`)



(a) Magnitude Open Line for varying Δt



(b) Magnitude Shorted Line for varying Δt

Figure 4: Magnitude Responses

where

$$N = r\left(2e^{2j\omega\Delta t} - 2e^{j\omega\Delta t}\right)e^{j\omega\tau_{int}} - 2e^{j\omega\tau_{int} + 2j\omega\Delta t} \tag{16}$$

$$D = r^2\left(e^{2j\omega\Delta t} - 2e^{j\omega\Delta t} + 1\right) + r\left(2e^{j\omega\Delta t} - 2e^{2j\omega\Delta t}\right) + e^{2j\omega\tau_{int} + 2j\omega\Delta t} + e^{2j\omega\Delta t} \tag{17}$$

The magnitude responses were plotted in Fig.4(a) and Fig.4(b), while Fig.5 shows the phase response for the open line. The shorted case has the same response, save for inverted phase. The simulation parameters were $\tau = 50\ \mu$s and Δt was varied between 5 μs and 50 μs. The peak of the response is at 5000 Hz, which is as expected for a $\frac{\lambda}{4}$ open/shorted section of line. In the plots, exact results were obtained when Δt was equal to 5 μs and 50 μs. In these cases, the line travelling time τ was an integer multiple of the simulation time step and the interpolated model reverts to the ideal case. In other cases, the influence of the interpolation becomes visible as a deviation at higher frequencies due to the reduced bandwidth.

It can be seen that for 20 μs, which is 2.5 times smaller than τ, the deviation is much less than for 35 μs, which is 1.43 times τ. This observation will be used later in this paper to define an interpolation-error criterion.

When using interpolation, it can be readily appreciated that the error will be highest when $r = 0.5$. For this value, the interpolated value is equally distant from the two known values in the history table. The effect of the r

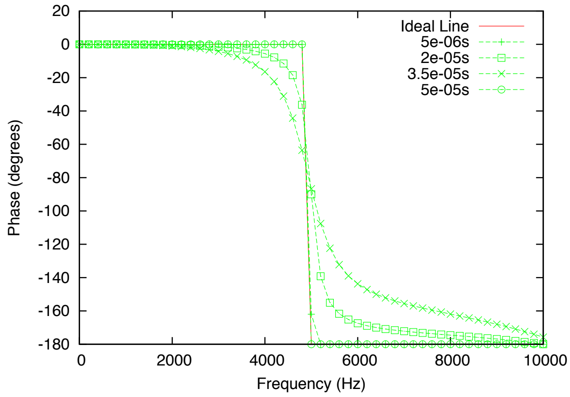


Figure 5: Phase Response Open Line for varying Δt

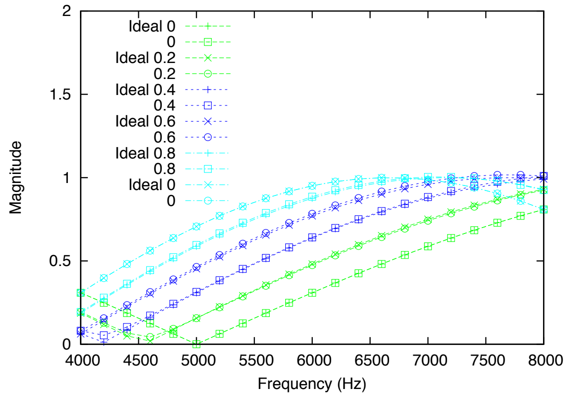


Figure 6: Magnitude Shorted Line for varying r

value on the simulation precision is illustrated by Fig.6. In these plots, the travelling time of the line was varied so that $0 \leq r < 1$. Δt was set to 25 μs and τ varied from 50 μs to 75 μs for the shorted case. The plotted curves are arranged so that the the top left traces correspond to an r of 1 (which is shown in the table as 0, as the r calculation will wrap around) and the bottom right traces for an r of 0.

As can be seen, only a small inaccuracy results from where in the interval the interpolated value is obtained, compared to the ideal case for each r. Do note that, as the travelling time is increased, the resonance frequency shifts down.

5 Interpolation Error Criterion

Since interpolation is only accurate when the changes between the discrete history values are small, the simulation time step has to be small enough compared to the wave speed on the line so that the rate of change in one time step is limited. This requirement is due to the low-pass filter introduced by the interpolation, which is an averaging procedure. Hence, inaccuracies occur for fast-changing signals due to the attenuation of higher-frequency components. A common error criterion used in the EMTP [6, 13] is to enforce a time step of five to ten times the line travelling time τ.

In addition, spatial and temporal Nyquist criteria must be satisfied. The simulation time step must be at most half the maximum frequency present in the simulation [16]. In practice, five to ten times the maximum frequency is recommended. This is the temporal Nyquist criterion. Since space is also discretised in matrix methods, a similar consideration applies to the size of the mesh elements compared to the wavelength of the travelling wave in the medium. To avoid these dispersion errors, the cells should be five to ten times smaller than the shortest wave length in the simulation [2, 3]. These simulation criteria can be summarised as follows:

EMTP

- Interpolation Criterion: $\Delta t_{max} = \frac{\tau_{max}}{10}$
- Nyquist Criterion: $\Delta t_{max} = \frac{1}{10 f_{max}}$

TINA

- Spatial Discretisation Criterion: $\Delta l_{max} = \frac{\lambda_{min}}{10}$
- Temporal Discretisation Criterion: $\Delta t_{max} = \frac{1}{10 f_{max}}$

Note that in TINA [7] and some TD-FD methods [5], Δl, the size of the cells in the mesh, is twice the length of the actual lines in the cell. Thus, when a criterion of five is used, it is in effect one of 10.

By combining these criteria, a set of guidelines for the simulation parameters, the maximum mesh cell size Δl_{min} and maximum simulation time step Δt_{max}, can be obtained (with v_{min} the wave speed of the slowest material and f_{max} the highest frequency in the simulation):

- Spatial Discretisation Criterion: $\Delta l_{max} = \frac{v_{min}}{10 f_{max}}$
- Temporal Discretisation Criterion: $\Delta t_{max} = \frac{1}{100 f_{max}}$

Thus, the use of interpolation requires a simulation time step one hundred times the highest frequency in the simulation compared to ten times for a non-interpolated case. Furthermore, for each line, ten memory positions are used for each variable in the interpolated line model. Compared to the single memory position used for each line in TLM-style simulations with matching stubs, the memory use can thus become an order of magnitude larger using the interpolated line.

It must be noted that the traditional stub-line method also suffers from a parasitic low-pass filter. The stub line forms a shunt capacitance (or inductance) and, with the line impedances, thus forms a filter as well. Thus, the normal condition of Δt ten times f_{max} would have to be increased due to the reduced bandwidth [3].

6 Time-Domain Response

One of the disadvantages of stubs is the parasitic oscillations in the response that occur due to the reflections on the stub [3]. When using interpolation, these are not present. Fig.7 illustrates the behaviour of two interpolated line cells connected in series, when excited with a

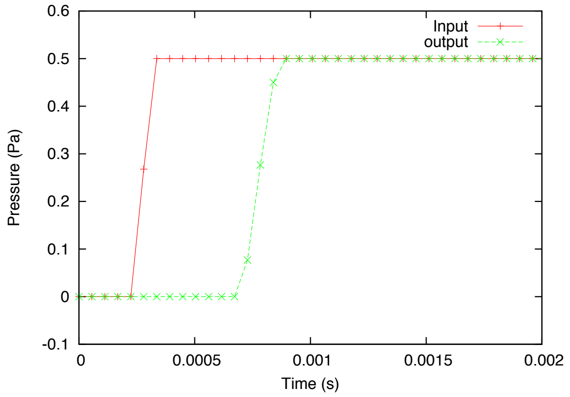


Figure 7: Time domain response to a step function

step function in TINA. Each line cell was constructed from two half-length sections. Both the source and load are matched to the lines. The signal data was measured on the centre node of the first cell (input) and on the centre node of the second cell (output). As such, there were two sections of line between the points of observation. The simulation time step was 56 μs and both lines had a τ of 250 μs. The impedance was 100 Ω. These parameters result in an r of 0.46, and 4.46 time steps per line.

As can be seen, the response does not exhibit any parasitic behaviour except for a reduced bandwidth. This reduction is made especially apparent though the use of a step, which theoretically requires infinite bandwidth. Moreover, a worst-case scenario for the interpolation accuracy was chosen, as the deviation from one time step to the next was the total signal amplitude and r was approximately 0.5.

7 Conclusions

In this paper, the use of interpolation is introduced to matrix methods, such as TLM and TD-FD. This conceptually simple approach proved sufficiently accurate and offers a better transient response than the traditional stub-matched approach to synchronisation. The method does require more memory and stricter time-step constraints than stub-matching.

The performance of the method was evaluated in comparison to the ideal line model and initial criteria for accuracy were obtained.

Future work will focus on the behaviour of the method in 2D and 3D and explore methods to reduce the memory consumption.

References

[1] P. Johns and R. Beurle, "Numerical solution of 2-dimensional scattering problems using a transmission line matrix," *Proceedings of the IEE*, vol. 188, no. 9, pp. 1203–1208, 1971.

[2] C. Christopoulos, *The Transmission-Line Modeling (TLM) Method in Electromagnetics*, ser. Synthesis Lectures on Computational Electromagnetics, C. Balanis, Ed. USA: Morgan & Claypool Publishers, 2006, no. 7.

[3] D. de Cogan, *Transmission Line Matrix (TLM) Techniques for Diffusion Applications.* Amsterdam, The Netherlands: Gordon and Breach Science Publishers, 1998.

[4] K. Yee, "Numerical solution of initial value problems involving maxwell's equations in isotropic media," *IEEE Transactions on Antennas and Propagation*, vol. 14, no. 3, pp. 302–307, May 1966.

[5] Z. Chen, M. Ney, and W. Hoefer, "A new finite-difference time-domain formulation and its equivalence with the tlm symmetrical condensed node," *IEEE Transactions on Microwave Theory and Techniques*, vol. 39, no. 12, pp. 2160–2169, December 1991.

[6] H. W. Dommel, *EMTP Theory Book*, 2nd ed. Vancouver, British Columbia, Canada: Microtran Power Systems Analysis Corporation, 1996.

[7] T. De Rybel, J. R. Martí, and M. Hodgson, "Modelling of acoustic wave propagation using transient insular nodal analysis (tina)," in *19th International Congress on Acoustics*, no. COM-06-004. Madrid, Spain: ICA'07, 2-7 September 2007.

[8] H. W. Dommel, "Digital computer solution of electromagnetic transients in single- and multiphase networks," *IEEE Trans. on Power Apparatus and Systems*, vol. PAS-88, no. 4, pp. 388–399, April 1969.

[9] P. Magnusson, *Transmission Lines and Wave Propagation.* Boston, USA: Allyn and Bacon, 1965.

[10] L. Kinsler, A. Frey, A. Coppens, and J. Sanders, *Fundamentals of Acoustics*, 3rd ed. USA: John Wiley & Sons, 1982.

[11] D. Christensen, *Ultrasonic Bioinstrumentation.* USA: John Wiley and Sons, 1988.

[12] L. Bergeron, *Water Hammer in Hydraulics and Wave Surges in Electricity.* New York: John Wiley & Sons, 1961.

[13] J. R. Martí, *"Constant-Parameters Ideal Line Model", EECE560 Class Notes.* Vancouver, British Columbia, Canada: The University of British Columbia, 1994.

[14] E. Kreyszig, *Advanced Engineering Mathematics*, 4th ed. USA: John Wiley & Sons, 1979.

[15] A. I. Ibrahim, S. Henschel, A. C. Lima, and H. W. Dommel, "Applications of a new emtp line model for short overhead lines and cables," *Electrical Power and Energy Systems*, vol. 24, no. 8, pp. 639–645, October 2002.

[16] A. V. Oppenheim, R. W. Schafer, and J. R. Buck, *Discrete-Time Signal Processing*, 2nd ed. New Jersey, USA: Prentice Hall, 1999.