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## Is auditory intensity discrimination a comparison of entropy changes?

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McConville, Norwich, & Abel hypothesized that two-alternative forced-choice discrimination of stimuli of identical spectra and durations, but differing intensities, depends upon the change in the information-theoretic entropy over the intensity difference and the duration. They tested their notion by devising a Weber-fraction equation in five unknowns, one being the entropy change. They presumed values for two of the unknowns. To obtain the others, the equation's finite changes were replaced by infinitely small differentials, and the approximation was curvefitted to one listener's Weber fractions. A value for the entropy change emerged, which was substituted back into the original Weber-fraction equation. The latter was then curvefitted to the same data, quantifying the 2 remaining unknowns and altogether "verifying" the entropy-change hypothesis. Unfortunately, the McConville et al. curvefitting is forced and flawed. The model covers some single, unspecifiable forced-choice trial, that need not yield the Weber fraction. None of the equation's unknowns can legitimately be presumed. Also, McConville et al. covertly assumed Fechner's postulate, now largely abandoned; and their replacement of finite differences by differentials actually made all theoretical Weber fractions subzero, an impossibility. All this also casts doubt on a 1995 paper by Wong & Norwich, who used the McConville et al. Weber-fraction equation.

## 1. Discriminability and the Weber fraction

Given a base intensity  $I$  and the just-noticeable intensity difference (jnd)  $\Delta I$ , where by convention  $\Delta I > 0$ ,

$$\text{the Weber fraction} = \frac{\Delta I}{I},$$

hence the intensity-difference limen (DL)

$$= 10 \log_{10} \left( 1 + \frac{\Delta I}{I} \right). \quad (1)$$

## 2. Auditory discriminability: the McConville, Norwich, & Abel derivation

McConville, Norwich, & Abel [1] extended Norwich's Entropy Theory of Perception (1975-present) to the case of auditory intensity discrimination. A later paper in *JASA* [2] by Wong & Norwich had much in common with McConville et al. Both papers were concerned with loudness; Wong and Norwich explicitly, McConville et al. implicitly (as the percept used to make two-alternative forced-choices; see below). Both used the Entropy Equation that arose in the Entropy Theory; Wong and Norwich constructed it, whereas McConville et al. merely assumed it. For some intensity change  $\Delta I$  at intensity  $I$ , Wong and Norwich explicitly equated  $(\Delta I)/I$  to the empirical Weber fraction, whereas McConville et al. assumed implicit equivalence. Both papers (Wong & Norwich explicitly, McConville et al. implicitly) assumed Fechner's postulate, that the jnd at any intensity corresponds to a fixed change in sensation. Both papers depended crucially upon curvefitting of equations to data to yield values of unknowns. Finally, both papers sought analytical solutions to their algebra by replacing finite differences by differentials.

The particular aim of McConville et al. [1] was to account for the empirical Weber fraction, observed to progressively decrease at moderate intensities and then increase at high intensities. Their work is synopsisized in the following section.

## 2.1 The Entropy Theory of Perception

In the Entropy Theory the sensory receptor "takes successive samples" of the "displacement of endolymph adjacent to a hair cell in the ear" [1, p158]. The endolymph displacement was assumed to fluctuate microscopically. McConville et al. represented the displacement variance (called the "variance of the stimulus") by  $\sigma_s^2$  and the mean displacement (called the "mean intensity" of the stimulus) by  $I$ . They then assumed

$$\sigma_s^2 = \beta_1 I^n \quad (2)$$

where  $n$  was "Stevens' Index". Similarly, perceived loudness and the firing rate of the neuron (in spikes/s) were assumed to relate not, as usual, to  $I$ , but to the *uncertainty regarding  $I$* . That "stimulus uncertainty" was the stimulus entropy, called  $H$ , given by a "variation of a Shannon entropy function" [3, p536]. The number of samples the receptor took was called  $m$ , where  $m \geq 1$ . McConville et al. described the derivation of  $H$  in detail and it can be found in numerous earlier and later Entropy Theory publications. In deriving  $H$ , infinities occurred in the algebra. Norwich and coauthors avoided them by introducing a "reference signal" having variance  $\sigma_R^2$  [4,5]. Altogether,  $H$  in "natural units" (i.e. logarithm to base  $e$ ) is

$$H = \frac{1}{2} \ln \left( 1 + \frac{\sigma_s^2}{m \sigma_R^2} \right) \quad (3)$$

[4,5]. For stimulus duration  $t$ , and "with the assumption of a constant neural sampling rate" [1, p159], sample size is  $m = \beta_2 t$ . McConville et al. define  $\beta = \beta_1 / (\beta_2 \sigma_R^2)$ , hence

*stimulus entropy [stim. information]  $H$*

$$= \frac{1}{2} \ln \left( 1 + \frac{\beta I^n}{t} \right); \quad (4)$$

$I = \text{intensity}, t = \text{duration}; \beta, n \text{ are unknowns}.$

## 2.2 Change of entropy and the two-interval forced-choice procedure for obtaining the intensity-difference limen

McConville et al. [1] stated that the Entropy Theory had been developed for a single steady stimulus, but that it could describe intensity discrimination in the two-alternative forced-choice (2AFC) task. In the case discussed by McConville et al., the listener is presented with a sequence of pairs of stimuli, each pair having the same frequency and duration. One stimulus of each pair is the base stimulus, of constant intensity within a listening session. The other stimulus, the comparison, is of different intensity, which may or may not vary during a given session, depending upon which of two specific 2AFC methods (see below) is used. Which of base or comparison appears in a given interval (first or second) is randomized. After hearing both intervals, the listener indicates the more intense stimulus, operationally by choosing the louder one. The procedure is repeated, covering some predetermined number of pairs. Figure 1 illustrates the stimuli.

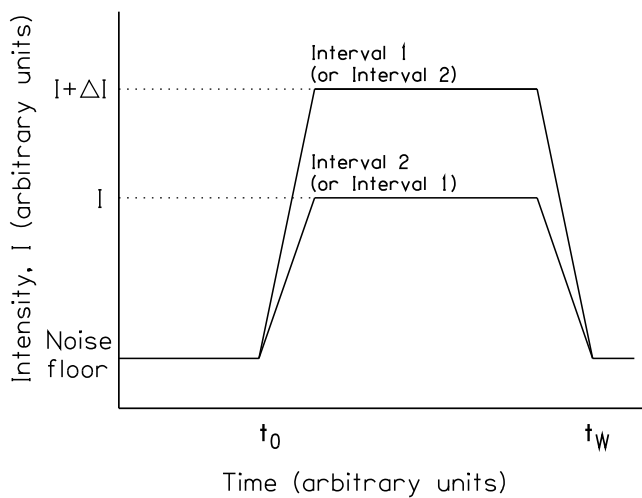


Figure 1. The intensity profiles for the stimuli used by McConville et al. [1] in a given trial of a typical 2AFC procedure, here shown separated by the intensity difference  $\Delta I$  to which the McConville et al. model applies (see later in text for details).

McConville et al. [1, p161] note that in the Entropy Theory, “The signal containing more information is the signal of greater intensity”. Hypothetically,  $H$  drops over the stimulus’ duration, just like perceived magnitudes in man and neuronal firing rates in animals, in the phenomenon called adaptation. Hypothetically, the most intense stimulus produces the largest drop in  $H$ , and *for that reason* is sensed as the loudest. The difference between the larger and smaller drop in  $H$  was expressed by McConville et al. by first obtaining the change in  $H$  for a single stimulus, over the span of “the time at which sensory neurons become activated after onset of the stimulus”, called  $t_0$ , to “the time at which the neural firing rate has adapted to its minimum value”, called  $t_w$  [1, p162-163]. That change, denoted  $\Delta H_t$ , is (from Eq.(4))

$$\Delta H_t = \frac{1}{2} \ln \left( 1 + \frac{\beta I^n}{t_0} \right) - \frac{1}{2} \ln \left( 1 + \frac{\beta I^n}{t_w} \right), \quad (5)$$

where  $\Delta H_t > 0$ .

The difference in  $\Delta H_t$  of stimuli of intensities  $I + \Delta I$  and  $I$  is  $\Delta H_t(I + \Delta I) - \Delta H_t(I)$ , denoted  $\Delta H_{I,t}$ , which when rearranged expresses the Weber fraction in terms of Entropy Theory parameters:

$$\Delta H_{I,t} = \frac{1}{2} \ln \left[ \frac{\left( 1 + \frac{\beta(I+\Delta I)^n}{t_0} \right) \left( 1 + \frac{\beta I^n}{t_w} \right)}{\left( 1 + \frac{\beta(I+\Delta I)^n}{t_w} \right) \left( 1 + \frac{\beta I^n}{t_0} \right)} \right], \quad (6)$$

from which

$$\frac{\Delta I}{I} = \left[ \frac{\frac{e^{2\Delta H_{I,t}} - 1}{I^n} + \frac{\beta e^{2\Delta H_{I,t}}}{t_0} - \frac{\beta}{t_w}}{\frac{\beta}{t_0} + \frac{\beta^2 I^n}{t_0 t_w} - \frac{\beta e^{2\Delta H_{I,t}}}{t_w} \left( 1 + \frac{\beta I^n}{t_0} \right)} \right]^{\frac{1}{n}} - 1. \quad (7)$$

Figure 2 illustrates these concepts. Eq.(7) has five unknowns:  $\Delta H_{I,t}$ ,  $\beta$ ,  $t_0$ ,  $t_w$ , and  $n$ .

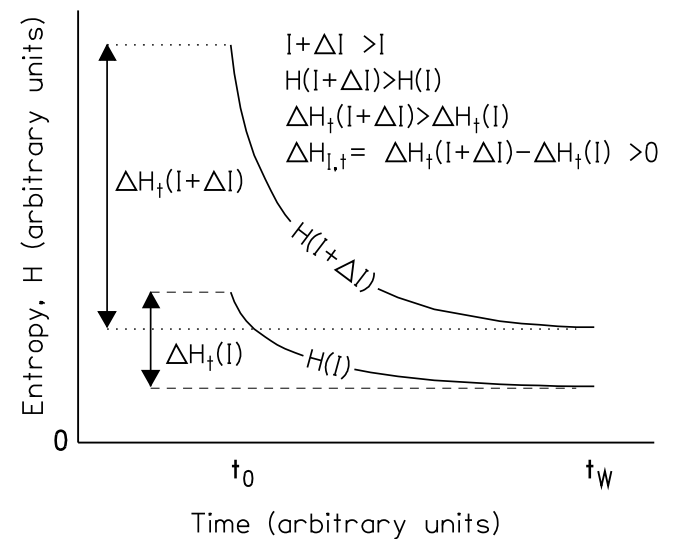


Figure 2. Entropy changes for a steady auditory stimulus, according to McConville et al. [1]. Over the time interval starting at  $t_0$ , “the time at which sensory neurons become activated after onset of the stimulus”, and  $t_w$ , “the time at which the neural firing rate has adapted to its minimum value”, a stimulus of intensity  $I$  causes an entropy change  $\Delta H_t(I)$ , and a stimulus of greater intensity  $I + \Delta I$  causes an entropy change  $\Delta H_t(I + \Delta I)$ . The difference in those entropy changes is the positive quantity  $\Delta H_{I,t}$ .

### 2.3 Empirical intensity-difference limens

McConville et al. [1] obtained Weber fractions from three subjects for 1-kHz, 300-ms tones over 10-90 dB SPL in 10-dB steps. They used the method of constant levels (see below)

using a score of 75% for the DL. Only one subject showed a notable DL *increase*, over 80-90 dB SPL.

## 2.4 Matching theory to data

McConville et al. [1] wished to match Eq.(7) to that one subject's data. First, they reduced the number of unknowns from five to two, as follows. The unknown  $\Delta H_{I,t}$ , for reasons explained below, "is regarded as a constant parameter to be estimated from the data" [1, p164], by fitting an approximation of Eq.(7) to the one subject's empirical Weber fractions:

$$\frac{\Delta H_{I,t}}{\Delta I \Delta t} \approx \frac{\partial^2 H}{\partial I \partial t} \quad (8a)$$

hence

$$\frac{\Delta I}{I} \approx \frac{\Delta H_{I,t}}{\frac{\partial^2 H}{\partial I \partial t} I \Delta t} \quad (8b)$$

where

$$\frac{\partial^2 H}{\partial I \partial t} = \frac{-n\beta}{2t^2} \left( I^{-\frac{1-n}{2}} + \frac{\beta I^{\frac{1+n}{2}}}{t} \right)^{-2}. \quad (9)$$

(Note that Eqs.(8a) and (8b) appeared in [6] as Eqs.(18) and (19) respectively.) McConville et al. [1, p162] then let  $\Delta t = t$  "as an approximation". They also stated that "the absolute value has been taken since  $\Delta I$  can be an increment or a decrement in intensity" [1, p162]. Altogether then,

$$\frac{\Delta I}{I} \approx \left| \frac{2 \Delta H_{I,t}}{n (\beta/t)} \right| \left( I^{-\frac{n}{2}} + \left( \frac{\beta}{t} \right) I^{\frac{n}{2}} \right)^2 \quad (10)$$

(called  $g(n,(\beta/t),\Delta H(I,t))$  in the published abstract; listed as Eq.(20) in [6]). McConville et al. then assumed that  $n$  ("Stevens' Index") = 0.25 (for the "best results"), rather than 0.3 (in [7]) or 0.33 (in [8]). They also assumed  $t_w = 0.020$  s, after [9] and [10]. Values for  $\Delta H_{I,t}$  and for  $\beta/t$  were obtained by fitting Eq.(10) to the data of the same single subject.

$\Delta H_{I,t}$  was 0.00653 (in "natural" units, i.e. using logarithms to base "e").  $\beta/t$  was tabulated, and then they ignored it.<sup>1</sup>  $n = 0.25$ ,  $t_w = 0.020$  s, and  $\Delta H_{I,t} = 0.00653$  were then substituted into Eq.(7), which was then fitted to the same data. The fit gave  $\beta = 0.000382$ , and  $t_0 = 0.00597$  s. The fitted curve was smoothly bowl-shaped, which McConville et al. took as validation of their model.<sup>2</sup>

## 3. Interpreting the McConville et al. derivation: (1) Relating entropy change to

## two-interval forced-choice

There are a number of problems with the McConville et al. [1] treatment of  $(\Delta I)/I$ . First, McConville et al. never explained how comparing entropy changes actually relates to 2AFC laboratory procedures; the listener's laboratory experience was treated as a *single* comparison, although actual 2AFC involves many comparisons.<sup>3</sup>

## 4. Interpreting the McConville et al. derivation: (2) The derived equation for $(\Delta I)/I$

The Entropy Theory does not *generate* any values for the term  $(\Delta I)/I$ , because all of  $\beta$ ,  $t_0$ ,  $t_w$ ,  $\Delta H_{I,t}$ , and  $n$  remain unknown.<sup>4</sup> As Norwich noted of the Entropy Theory near the time of its conception, "The model was rather arbitrary and served to crystallize the qualitative relationships" [4, p611]. That is, Eq.(4) was pure conjecture, and so are its successors, Eqs.(5), (6), (7), and (10). Thus any agreement of Eq.(7) with empirical values of  $(\Delta I)/I$  is coincidental. The  $\Delta I$  in all the McConville et al. equations can be *any* intensity change; it is not uniquely that of the empirical Weber fraction. Thus, the stimulus-entropy-change associated with the jnd can only be obtained by matching the former to the latter, which is what McConville et al. actually did, as the Entropy Theory equations contain unknowns that could not be obtained in any other way.

## 5. Interpreting the McConville et al. derivation: (3) The constancy of the change in stimulus entropy

McConville et al. had declared  $\Delta H_{I,t}$  to be constant. They did not state why. The present author found the reason hidden in [6] (p173, footnote). There, it was hypothesized that sensation, which Norwich denoted  $F$ , is related to  $H$  through  $F = kH$ , where  $k$  is a constant for a given situation. Norwich [6] then suggested that we adopt Fechner's postulate, that the jnd at any intensity corresponds to a fixed change in sensation:  $\Delta F = \text{constant}$ . But  $\Delta F = \Delta(kH) = k \Delta H_{I,t}$ , so that  $k \Delta H_{I,t} = \text{constant}$ . With  $k$  already constant,  $\Delta H_{I,t}$  was constant. However, Fechner's postulate is no longer accepted for the larger part of the auditory intensity range (e.g. [11,12]).

## 6. Interpreting the McConville et al. derivation: (4) The use of the approximation to the entropy changes

$n$ ,  $\beta$ , and  $t$  are, by their definitions, all positive numbers. Thus Eq.(9) returns a *negative* number. Now,  $\Delta I$  is defined in the literature as positive, consistently.  $\Delta t$  and  $I$  are always positive. Altogether, for Eq.(8b) to return  $(\Delta I)/I > 0$ , then  $\Delta H_{I,t} < 0$ . But it is provable (Appendix, below) that  $\Delta H_{I,t} > 0$  if  $\Delta I > 0$ . Altogether, Eq.(8b) only yields  $(\Delta I)/I < 0$ .

It seems fair to question how Eq.(9) yields a negative. Consider the origin of Eq.(8a). It appeared in [6] as “Eq.(18)” and was explained by Norwich [6, p173, footnote] as follows: “It must be cautioned, however, that the above [ie. “Eq.(18)”] is not a general approach to the study of approximate values of functions of two independent variables such as  $x(t,m)$  [sic], for which Taylor’s formula must be used” (bracketed terms supplied). Taylor’s formula [13, p439] gives the second differential of a function  $H = H(I,t)$  as

$$d^2 H(I,t) = \frac{\partial^2 H(I,t)}{\partial^2 I} (dI)^2 + 2 \frac{\partial^2 H(I,t)}{\partial I \partial t} dI dt + \frac{\partial^2 H(I,t)}{\partial^2 t} (dt)^2, \quad (11a)$$

hence approximately

$$\frac{\Delta^2 H(I,t)}{\Delta I \Delta t} = \frac{\partial^2 H(I,t)}{\partial^2 I} \frac{\Delta I}{\Delta t} + 2 \frac{\partial^2 H(I,t)}{\partial I \partial t} + \frac{\partial^2 H(I,t)}{\partial^2 t} \frac{\Delta t}{\Delta I}. \quad (11b)$$

$\Delta^2 H(I,t)$  is what McConville et al. call  $\Delta H_{I,t}$ . In Eq.(8a) McConville et al. used only the second term of Eq.(11b), and without the “2”. Thus they implicitly assumed that the 2 is ignorable. They also implicitly assumed that the first and third terms of Eq.(11b) are negligible compared to the second term. Recall that  $\Delta H_{I,t} > 0$  (Appendix); yet Eq.(9) returns a negative value, and hence a negative  $(\Delta I)/I$ . To avoid that, Eq.(8a) needs more terms of Eq.(11b).<sup>5</sup>

The flaws revealed in this section should clearly indicate that inferring the values of any parameters by the curvefitting of Eq.(10) is invalid. Hence, the number of free parameters remains five, too many for robust curvefitting of theory to data.

## 7. Summary and conclusions

The Entropy Theory of Perception of Norwich et al. was applied by McConville, Norwich, & Abel [1] to two-interval forced-choice auditory intensity discrimination. They derived an equation for  $(\Delta I)/I$  in five unknowns. Values for two of those unknowns were assumed *a priori*. The remaining three were quantified by curvefitting, in two stages, to a single subject’s empirical Weber fractions. Unfortunately, (1) McConville et al. cannot specify how their model integrates into actual forced-choice procedures; (2) their  $(\Delta I)/I$  equation is pure conjecture, i.e. values for none of the five unknowns can be specified *a priori*; (3) they covertly assume Fechner’s postulate, which, in the literature, is not accepted for the majority of the auditory intensity range; and (4) their *approximate*  $(\Delta I)/I$  is negative, due to an incomplete series expansion. Altogether, the McConville, Norwich, & Abel [1] paper is completely incorrect. A later paper by Wong &

Norwich [2] that was the conceptual successor to McConville, Norwich, & Abel [1] must consequently be viewed with caution.

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## Footnotes

1. Eq.(8a) led McConville et al. to Eq.(10) for  $(\Delta I)/I$ . Knowing  $t$  allows  $\beta$  to be found by curvefitting of Eq.(10) to empirical Weber fractions; but instead, McConville et al. treated  $\beta/t$  as a single unknown, so that  $t$  could not be separated from  $\beta$ . Nonetheless, their fitted value of  $\beta/t$  [1, Table 1], 0.032, can be multiplied by their empirical stimulus duration of  $t = 0.300$  s to give  $\beta = 0.0096$ . The latter is roughly 25 times the 0.000382 that McConville et al. later obtained for  $\beta$  by fitting Eq.(7) [1].

2. McConville et al., in their Table 1 [1], list the units of  $\beta$  as  $s^{-1}$ . They list  $\beta/t$  as unitless. In fact,  $\beta$  has units of (time)·(intensity) $^{-n}$ , thus  $\beta/t$  has units of (intensity) $^{-n}$ . In this context,  $n$  is required to be unitless.

3. Usually, 2AFC involves one of two methods. In the *method of constant levels*, an experimental session consists of repeated comparisons of a given, fixed comparison stimulus and the base stimulus, enough times to obtain a percentage correct. The intensity of the comparison stimulus is then changed, and the procedure repeated to get another percentage correct. A set of percentages is thus obtained, and is fitted to a monotonic function of comparison-stimulus intensity, usually a sigmoidal function. Then some criterion level of performance (such as 75% correct) is declared and the difference between the intensities of the comparison stimulus that corresponds to the criterion, and the base stimulus, is  $\Delta I$ . In contrast, there is the faster and hence more popular adaptive-tracking method, a kind of “staircase” method. There,  $\Delta I$  can be inferred based on a single listening session. The session starts with the comparison stimulus being unmistakable in intensity, then being progressively decremented or incremented from one pair of intervals to the next, respectively according to whether the subject responds correctly or not on the comparison (plus possibly one or more previous comparisons). The total number of comparisons in a listening session is set so that the intensity of the comparison stimulus eventually wavers back and forth between a mean value. That mean is subtracted from the base intensity to give  $\Delta I$ .

4. McConville et al. assigned a value to  $n$  on the unsupported *assumption* that it equalled Stevens’ Index. In fact a value cannot be specified for  $n$ , because substituting Eq.(2) into Eq.(3), and assuming further that loudness is proportional to  $H$ , does not give the simple power function which is called Stevens’ Law when applied to loudness growth. Other terms cannot be specified either. The term  $\beta = \beta_1/(\beta_2 \sigma_R^2)$  contains elements of three things: a hypothesized sampling process, an unknown internal noise, and the hypothesized relation of stimulus intensity variance to mean stimulus intensity (Eq.(2)). Thus, altogether,  $\beta$  cannot be known *a priori*.  $t_w$  and  $t_0$  cannot be assigned values unambiguously; more than fifty years of accumulated literature clearly indicates that (1) the adapting auditory firing rate in a primary afferent responding to a steady stimulus never reaches an unambiguous plateau, hence “the time at which the neural firing rate has adapted to its minimum value” cannot be specified, and (2) neither can “the time at which sensory neurons become activated after onset of the

stimulus”, because the latter is not distinct, and further, it is well-established that one possible indicator of  $t_0$ , the fast initial rise in firing rate to peak firing rate, shortens with rise in intensity (but  $t_0$  had to be assigned some nonzero value because  $t_0 = 0$  produced an infinite  $H$ ). Finally, the adopted value for  $t_w$ , 20 ms (taken from [9] and [10]), may be inappropriate; the recordings of Galambos and Davis [9] were taken not from primary afferents as Galambos and Davis had believed, but were in fact taken from “cell bodies of second-order neurons” [14, p513], casting doubt on [10] as well..

5. Eq.(8a) actually introduced a new unknown,  $\Delta t$ . McConville et al. removed it by setting  $t = \Delta t$ . But as McConville et al. noted, “The time,  $t$ , is constant for a given experiment, representing the duration of exposure to the stimulus” [1, p162; repeated in 6, p174]. Thus  $t$  is a known quantity, and its suitability as a substitute for  $\Delta t$ , whose magnitude is required to be small from Eq.(8a), could have been evaluated. McConville et al. declined to do so.

## Appendix: proof that $\Delta H_{I,t} > 0$

Let us assume that  $\Delta I > 0$  and that  $\Delta H_{I,t} \leq 0$ . Then from Eq.(6),

$$\frac{\left[ \left( 1 + \frac{\beta(I+\Delta I)^n}{t_0} \right) \left( 1 + \frac{\beta I^n}{t_w} \right) \right]}{\left[ \left( 1 + \frac{\beta(I+\Delta I)^n}{t_w} \right) \left( 1 + \frac{\beta I^n}{t_0} \right) \right]} \leq 1. \quad (A1)$$

Rearranging, multiplying through, and eliminating terms common to both sides leaves

$$\frac{(I+\Delta I)^n}{t_0} + \frac{I^n}{t_w} \leq \frac{(I+\Delta I)^n}{t_w} + \frac{I^n}{t_0}. \quad (A2)$$

Establishing the common denominator  $t_0 \cdot t_w$ , multiplying through by it, then collecting like terms gives

$$(I+\Delta I)^n (t_w - t_0) \leq I^n (t_w - t_0). \quad (A3)$$

That is impossible, as all terms are positive, and  $I+\Delta I > I$  by assumption. Thus  $\Delta H_{I,t} > 0$  when  $\Delta I > 0$ .