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Semi-empirical time domain model of sound attenuation in porous materials

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A semi-empirical model for complex tortuosity function, which satisfies physically correct low and high frequency limits and allows analytical transformation into the time domain has been developed. It is based on the assumption, that a network of pores with two characteristic sizes can approximate the internal structure of the material, and thus requires the knowledge of two relaxation times. It is proven, however, that the model can predict sufficiently well the acoustical properties of rigid porous materials with various microstructures when is complete with the tortuosity as an additional parameter. It is shown that relaxation times can be easily related to the "equivalent fluid" model parameters. It is shown that the extended version of the model which accounts for the thermal effects can also be used for time-domain computations.

1 Introduction

The equations for pressure p and particle velocity v variations in a plane wave of angular frequency ω can be written in the following form

$$-i\omega\rho_0\alpha\left(\omega\right)v = -\partial_x p,\quad -i\omega C\left(\omega\right)p = -c^2\rho_0\partial_x v,$$

where dimensionless complex density $\alpha\left(\omega\right)$ and complex compressibility $C\left(\omega\right)$ describe viscous and thermal effects in a porous medium. One of the most successful models for the acoustical properties of rigid porous materials has been suggested in [1] where physically correct solutions for the complex tortuosity function in the limiting cases of low and high frequencies have been formulated

$$\alpha\left(\omega\to 0\right) = \frac{\phi\sigma}{-i\omega\rho_0}, \qquad (1)$$

$$\alpha_\infty\left(\omega\to\infty\right) = \alpha_\infty + \sqrt{\frac{\nu}{-i\omega}\frac{2\alpha_\infty}{\Lambda}}.$$

Then a function,

$$\alpha(\omega) = \alpha_\infty + \frac{\sigma\phi}{-i\omega\rho_0}\sqrt{1 - i\omega\nu\left(\frac{2\rho_0\alpha_\infty}{\sigma\phi\Lambda}\right)^2}, \qquad (2)$$

which satisfactory interpolates between these limits was constructed and was proven to give reliable results for many rigid porous materials.

Similar approach to modelling complex compressibility function $C\left(\omega\right)$ has been later suggested in [2, 3] and resulted in the following approximation

$$C(\omega) = \gamma - (\gamma - 1)\left(1 + \frac{\nu\phi}{-i\omega N_{pr}k'}\sqrt{1 - \frac{i\omega N_{pr}}{\nu}\left(\frac{2k'}{\varphi\Lambda'}\right)^2}\right)^{-1} \qquad (3)$$

satisfying physically correct low and high frequency limits

$$C\left(\omega\to 0\right) = \gamma - \left(\gamma - 1\right)\frac{-i\omega N_{pr}k'}{\nu\varphi},$$

$$C\left(\omega\to\infty\right) = 1 + \left(\gamma - 1\right)\sqrt{\frac{\nu}{-i\omega N_{pr}}\frac{2}{\Lambda'}}. \qquad (4)$$

The combined model requires knowledge of flow resistivity σ, thermal permeability k', tortuosity α_∞, porosity ϕ and characteristic viscous and thermal lengths Λ and Λ'. They all, in principle, can be measured non-acoustically.

Another approach to the modelling has been developed in [4, 5] and is based on viewing viscous and thermal diffusion in porous media as a relaxation process. The model approximates the complex density function using only two parameters: tortuosity and characteristic viscous time τ. One more parameter has to be introduced to account for thermal effects. Unfortunately to satisfactorily approximate physically correct limits (1) the model would require different values for τ at low and high frequencies. The model extension, which includes two viscous relaxation times has been developed in the same publication which overcomes this difficulty.

The interest in time domain analogues of these models has grown recently in connection with the successful application of time domain computational models in atmospheric and room acoustics. When model for sound propagation in air is formulated in the time domain, common acoustic impedance boundary conditions for the ground or absorbing walls can not be used and the explicit formulation of the time domain model for the porous medium is required. Inverse Fourier transformation of the governing equations can be performed analytically only for specific functions $\rho(\omega)$ and $C(\omega)$. Functions (2) and (3) do not allow analytical conversion into the time domain for the whole range of frequencies. However in the lower frequency limit the conversion is possible [6]. The time domain formulation which is applicable to high frequencies (where inertial effects become stronger) has been developed recently [7] and relates to Basset history term for the viscous drag force. In terms of pulse duration those two versions describe very long or very short acoustic pulses (comprising of rarefaction and compression phases) respectively but not the pulses of intermediate duration for which both viscous and inertial effects are equally strong**.** Recently it was found [8] that the simplest version of the relaxation model can be easily converted into the time domain for all frequencies but due to its above mentioned limitations the description of all duration pulses with just one relaxation time is not possible. The version of the model with two relaxation times does not allow analytical transformation into the time domain.

In this work the problem of designing empirical complex tortuosity and complex compressibility functions which satisfy physically correct low and high frequency limits (1) and (4) and can be transformed analytically into the time domain for the whole range of frequencies is addressed.

2 Models for the networks of cylindrical pores

2.1. General time domain form of the momentum conservation equation

The most general form of complex tortuosity function which follows from [9] is

$$\alpha(\omega)=\frac{\alpha_\infty}{1-\sum_{n=1}^{\infty}\frac{a_n^{\;2}}{1-i\omega\Theta_n}}, \qquad (5)$$

where Θ_n are viscous relaxation times which are related to n-th eigenvalue ε_n of the time domain pore fluid flow problem as $\Theta_n=\frac{1}{\nu\varepsilon_n}$. Coefficients $a_n^{\;2}$ satisfy the following conditions: $\sum_{n=1}^{\infty}a_n^{\;2}=1,\ \ \sum_{n=1}^{\infty}a_n^{\;2}\Theta_n=\frac{\rho_0}{\alpha_\infty\phi\sigma}$.

Inverse Fourier transform of the momentum conservation equation with expression (5) for complex tortuosity leads to the following time domain equation

$$\alpha_\infty\rho_0\partial_t V=-\partial_x P+\sum_{n=1}^{\infty}\frac{a_n^{\;2}}{\Theta_n}\int_{-\infty}^{t}\partial_x P(t')\exp\left(-\frac{(t-t')}{\Theta_n}\right)dt', \qquad (6)$$

where

$$V(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}v(\omega)e^{-i\omega t}dt$$

$$P(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}p(\omega)e^{-i\omega t}dt .$$

Finding coefficients a_n and Θ_n analytically is possible for the very simple pore geometries only. However, even in those cases sums in (6) can be slowly converging and hence approximate solutions should be used.

2.1. Network of identical straight cylindrical pores – relaxation function

For the network of identical straight cylindrical pores $a_n=\frac{2}{j_n}$, $\Theta_n=\frac{4\tau}{j_n^{\;2}}$ and $\alpha_\infty=1$,

where $\tau=\frac{a^2}{4\nu}$, a is the pore radius and j_n is Bessel function zero of order n so that $J_0(j_n)=0$.

Substituting these values into (6) leads to the following

$$\rho_0\partial_t V=-\partial_x P+$$
$$\frac{1}{\tau}\int_{-\infty}^{t}\partial_x P(t')\sum_{n=1}^{\infty}\exp\left(-\frac{(t-t')}{\tau}\left(\frac{j_n}{2}\right)^2\right)dt' \qquad (7)$$

In frequency domain, complex tortuosity function for the network of straight cylindrical pores can be calculated analytically and is equal to

$$\alpha_{cyl}(\omega)=\frac{1}{1-\frac{1}{\sqrt{i\omega\tau}}\frac{J_1\left(2\sqrt{i\omega\tau}\right)}{J_0\left(2\sqrt{i\omega\tau}\right)}},$$

It has been shown [4] by direct comparison in the wide range of frequencies that this function can be reasonably well approximated by the following “relaxation function”

$$\alpha_\tau=\frac{\alpha_\infty}{1-\frac{1}{\sqrt{1-i\omega\tau}}} \qquad (8)$$

The use of this function leads to the following approximate time domain equation

$$\alpha_\infty\rho_0\partial_t V=-\partial_x P+\frac{1}{\sqrt{\pi\tau}}\int_0^t\partial_x P(t')\frac{\exp\left(-\frac{t-t'}{\tau}\right)}{\sqrt{t-t'}}dt' \qquad (9)$$

Comparison of the exact equation (5) with this one leads to the conclusion that approximation is based on the following replacement

$$f(x)=\sum_{n=1}^{\infty}\exp\left(-x\left(\frac{j_n}{2}\right)^2\right)\rightarrow g(x)=\frac{\exp(-x)}{\sqrt{\pi x}}$$

where $x=\frac{t-t'}{\tau}$ is positive.

Comparison of the two functions is shown in Figure1. They differ considerably only in the range of x where their absolute values are extremely small which means that their integrals are close to zero anyway. Consequently convolution integral in (9) provides a very good approximation to the exact time domain equation (7) of sound propagation in network of straight cylindrical pores.

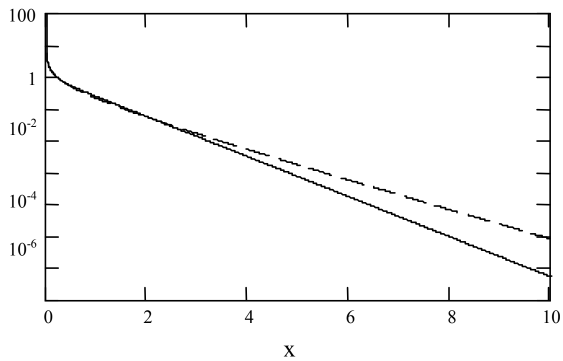

Fig.1 Comparison of functions f(x) and g(x)

Solid line – f(x) , i.e. exact solution, dashed line – g(x) , i.e. approximate solution.

Similarly to the complex tortuosity, a relaxation–type complex density function was introduced in [4],

$$C(\omega) = 1 + \frac{\gamma - 1}{\sqrt{1 - i\omega\tau_e}}, \tag{10}$$

which requires additional relaxation τ_e time and provides a good approximation to the exact solution of the heat transfer problem in the cylindrical pore with $\tau_e = N_{pr}\tau$. As shown in [8] the use of this function allows time domain formulation of the continuity equation:

$$\partial_t p + \frac{\gamma - 1}{\sqrt{\pi\tau_e}}\int_{-\infty}^{t}\frac{\partial_{t'} p(t')}{\sqrt{t - t'}}\exp\left(-\frac{t - t'}{\tau_e}\right)dt' = -c^2\rho_0\partial_x v.$$

2.1. Two types cylindrical pore networks–relaxation functions and time domain equations

Now let's consider a network of straight cylindrical pores with radii a_m, where m=1, 2…M and relative porosities ϕ_m, so that $\sum_{m=1}^{M}\phi_m = \phi$ is the total porosity of the material.

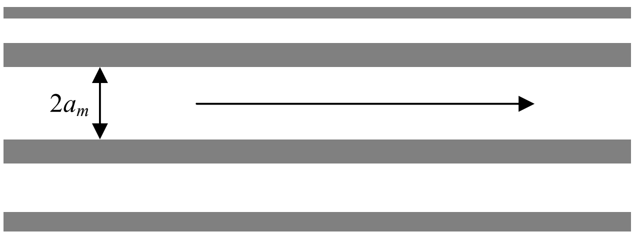

Fig.2 Straight pore network consisting of different size pores

According to [10] in this case the permeability of the slab is the arithmetic average of the permeabilities of each network. In terms of complex tortuosities this can be reformulated as

$$\alpha_{||}(\omega) = \frac{\alpha_\infty}{1 - \frac{1}{\phi}\sum_{m=1}^{M}\frac{\phi_m}{\sqrt{1 - i\omega\tau_m}}} \tag{11}$$

where each network tortuosity was approximated with relaxation function (7) with $\tau_m = \frac{a_m^{\ 2}}{4\nu}$. The complex compressibility functions can be formulated using the relationship with complex density valid for the cylindrical pores [11] and result in the following

$$C_{||}(\omega) = \left(\gamma - (\gamma - 1)\frac{\alpha_\infty}{\alpha_{||}(\omega N_{pr})}\right). \tag{12}$$

This result is in line with the argument mentioned earlier as complex compressibility is proportional to dynamic thermal permeability and hence has to be calculated as arithmetic average. The inverse Fourier transform of the momentum and mass conservation equations leads to the following set of equations for the network in consideration:

$$\alpha_\infty\rho_0\partial_t V = -\partial_x P + \frac{1}{\varphi\sqrt{\pi}}\int_{-\infty}^{t}\frac{\partial_x P(t')}{\sqrt{(t - t')}}\sum_{m=1}^{M}\frac{\varphi_m\exp\left(-\frac{t - t'}{\tau_m}\right)}{\sqrt{\tau_m}}dt' \tag{13}$$

$$\partial_t P + \frac{(\gamma - 1)}{\sqrt{\pi}\phi}\int_{-\infty}^{t}\frac{\partial_{t'}P(t')}{\sqrt{t - t'}}\sum_{m=1}^{M}\frac{\phi_m\exp\left(-\frac{t - t'}{\tau_m N_{pr}}\right)}{\sqrt{\tau_m N_{pr}}}dt' = -c^2\rho_0\partial_x V$$

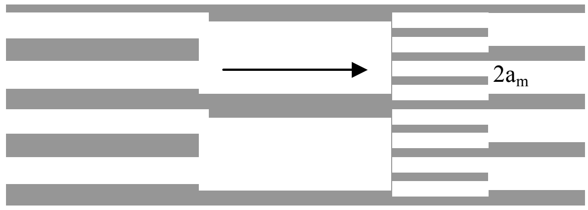

Fig.3 Straight pore network consisting of layers with different pose sizes and porosities

The effective permeability of the composite material shown in Figure 3 should be calculated as harmonic average of the phase permeabilities [10]. Reformulation in terms of tortuosity and the use of relaxation function approximation leads to the following

$$\alpha_\perp(\omega) = \frac{\alpha_\infty}{\phi}\sum_{m=1}^{M}\frac{\Omega_m\phi_m}{\left(1 - \frac{1}{\sqrt{1 - i\omega\tau_m}}\right)} \tag{14}$$

where the total porosity of the material is now calculated as $\phi = \sum_{m=1}^{M}\Omega_m\phi_m$ and Ω_m is volume fraction of the composite occupied by each layer. The resulting expression (14) is in fact quite similar to the "sectionally uniform tube model" described in [12]. According to the arguments mentioned above, the complex compressibility function follows the same summation law as permeability and is equal to

$$C_\perp(\omega) = \gamma - (\gamma - 1)\phi\left(\sum_{m=1}^{M}\frac{\phi_m\Omega_m}{1 - \frac{1}{\sqrt{1 - i\omega N_{pr}\tau_m}}}\right)^{-1} \tag{15}$$

This result is different from [12], the reasons for this are being investigated.

Following algebraic manipulations described in [8] the time domain equation can be now derived for the network shown in Figure 1

$$\partial_t V + \frac{V}{\phi}\sum_{m=1}^{M}\frac{\Omega_m \phi_m}{\tau_m} + \frac{1}{\phi}\sum_{m=1}^{M}\frac{\Omega_m \phi_m}{\sqrt{\pi\tau_m}} I_m = -\frac{\partial_x P}{\alpha_\infty \rho_0},$$

(16)

$$(\gamma-1)\partial_t P = \partial_t Q + \frac{Q}{\phi}\sum_{m=1}^{M}\frac{\phi_m \Omega_m}{N_{pr}\tau_m} + \frac{1}{\phi}\sum_{m=1}^{M}\frac{\phi_m \Omega_m}{\sqrt{\pi N_{pr}\tau_m}} J_m,$$

where

$$I_m = \int_{-\infty}^{t}\frac{V(t')/\tau_m + \partial_{t'}V(t')}{\sqrt{t-t'}}\exp\left(-\frac{t-t'}{\tau_m}\right)dt',$$

$$J_m = \int_{-\infty}^{t}\frac{Q(t')/N_{pr}\tau_m + \partial_{t'}Q(t')}{\sqrt{t-t'}}\exp\left(-\frac{t-t'}{N_{pr}\tau_m}\right)dt'$$

and $Q = \gamma P + \rho_0 c^2 \partial_x V$.

3 Model generalization for the arbitrary geometry rigid porous materials

Here the simple semi-empirical models allowing analytical time domain representation will be introduced based on the relaxation functions derived for the cylindrical pore networks in Section 2.

Consider the following function

$$\alpha_1(\omega) = \frac{\alpha_\infty}{1 - \frac{1}{2}\left(\frac{1}{\sqrt{1-i\omega\tau_1}} + \frac{1}{\sqrt{1-i\omega\tau_2}}\right)}. \tag{17}$$

It can be seen as a simplification of (11) assuming network of pores with two different sizes and $\phi_1 = \phi_2 = 0.5\phi$. It can be easily shown that the use of two relaxation times

$$\tau_{1,2} = \frac{2\alpha_\infty \rho_0}{\sigma\phi}\left(1 \pm \sqrt{1 - \left(\frac{1+\sqrt{8M+1}}{4M}\right)^2}\right), \tag{18}$$

where

$$M = \frac{8\alpha_\infty \eta}{\sigma \Lambda^2 \phi} \tag{19}$$

is the shape factor, allows matching of both correct low and high frequency limits (1) provided that $M \geq 1$.

Similarly, the complex compressibility function, introduced as a simplified version of $C_{||}(\omega)$

$$C_1(\omega) = 1 + \frac{\gamma-1}{2}\left(\frac{1}{\sqrt{1-i\omega\tau_{e,1}}} + \frac{1}{\sqrt{1-i\omega\tau_{e,2}}}\right), \tag{20}$$

satisfies limits (4) with the following pair of relaxation times

$$\tau_{e,1,2} = \frac{2k' N_{pr}}{\nu\phi M'}\left(1 \pm \sqrt{1 - \frac{\sqrt{8M'+1}-1}{2M'}}\right)^{-2}, \tag{21}$$

when $M' \geq 1$ and the thermal shape factor is defined as

$$M' = \frac{8k'}{\phi \Lambda'^2}. \tag{22}$$

Another function, based on the complex tortuosity (14) for the material shown in Figure 2 assuming two layers with the same values of $\Omega_m \phi_m$,

$$\alpha_2(\omega) = \frac{\alpha_\infty}{2}\left(\frac{1}{1 - \frac{1}{\sqrt{1-i\omega\tau_1}}} + \frac{1}{1 - \frac{1}{\sqrt{1-i\omega\tau_2}}}\right) \tag{23}$$

also matches limits (1) with

$$\tau_{1,2} = \frac{2\alpha_\infty \rho_0}{\sigma\phi\left(1 \pm 2\sqrt{M(1-M)}\right)} \tag{24}$$

provided that $0.5 < M \leq 1$.

Similarly, the expression for the complex compressibility function can be derived from (15) as follows

$$C_2(\omega) = \gamma - 2(\gamma-1)\left(\frac{1}{1 - \frac{1}{\sqrt{1-i\omega\tau_{e,1}}}} + \frac{1}{1 - \frac{1}{\sqrt{1-i\omega\tau_{e,2}}}}\right)^{-1} \tag{25}$$

This function satisfies correct limits (4) with

$$\tau_{e,1,2} = \frac{2k' N_{pr}}{\nu\phi\left(1 \pm 2\sqrt{M'(1-M')}\right)}, \tag{26}$$

and requires $0.5 < M' \leq 1$.

It is worth noting that for M=0.5 one of the relaxation times (24) and (26) approaches infinity.

Application of both sets of functions, i.e. (17), (20) and (23), (25), results in analytical time domain momentum and continuity equations.

Functions (17) and (20) give

$$\alpha_\infty \rho_0 \partial_t V = -\partial_x P + \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{t}\frac{\partial_x P(t')}{\sqrt{t-t'}}\sum_{m=1}^{2}\frac{\exp\left(-\frac{t-t'}{\tau_m}\right)}{\sqrt{\tau_m}}dt'$$

(27)

$$\partial_t P+\frac{\gamma-1}{2\sqrt{\pi}}\int_{-\infty}^{t}\frac{\partial_{t'}P(t')}{\sqrt{t-t'}}\sum_{m=1}^{2}\frac{\exp\left(-\frac{t-t'}{\tau_{e,m}}\right)}{\sqrt{\tau_{e,m}}}dt'=-c^2\rho_0\partial_x V,$$

while (23) and (25) allow the following transformation

$$\partial_t V+\frac{V}{2}\sum_{m=1}^{2}\frac{1}{\tau_m}+\frac{1}{2}\sum_{m=1}^{2}\frac{I_m}{\sqrt{\pi\tau_m}}=-\frac{\partial_x P}{\alpha_\infty\rho_0},$$

(28)

$$\left(\gamma-1\right)\partial_t P=\partial_t Q+\frac{Q}{2}\sum_{m=1}^{2}\frac{1}{\tau_{e,m}}+\frac{1}{2}\sum_{m=1}^{2}\frac{J_m}{\sqrt{\pi\tau_{e,m}}},$$

whith convolution integrals I_m defined above and

$$J_m=\int_{-\infty}^{t}\frac{Q(t')/\tau_{e,m}+\partial_{t'}Q(t')}{\sqrt{t-t'}}\exp\left(-\frac{t-t'}{\tau_{e,m}}\right)dt'.$$

The values of shape factors M and M' depend on the pore scale geometry as well as on the material porosities. There are numerous experimental as well as numerical indications [13], that viscous shape factor M is greater than unity in granular materials resembling close packings of spheres and equations (27) should provide adequate time-domain description of sound propagation in these materials. At the same time, FEM predictions shown in Figure 4, confirm that viscous shape factor is smaller than one for the array of circular cylinders assuming that sound propagates perpendicular to their axes.

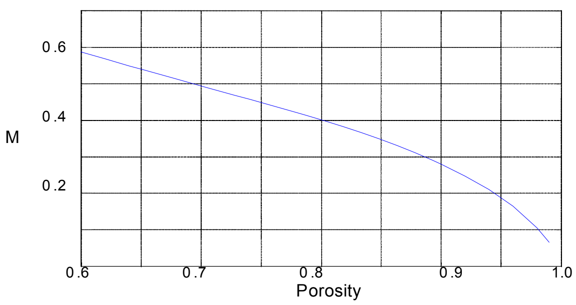

Fig.4 Dependence of viscous shape factor M on porosity, regular array of cylinders

This means that set of equations (28) could be applicable to relatively low porosity fibrous materials.

6. Conclusions

Two sets of semi-empirical time domain equations describing sound propagation in rigid porous materials have been suggested. The first set (27) is suitable for materials with viscous and thermal shape factors exceeding one, such as dense packings of spheres. In another set (28) it is assumed that both shape factors are smaller than one but still larger than 0.5. For each model two pairs of viscous and thermal relaxation times have been derived which match physically correct limits for complex tortuosity and complex density functions. This allows using the new time domain equations for the description of short and as well as long pulses propagating in porous materials. Comparisons with other model predictions for some types of granular materials as well as with the transmission data are presented in the companion paper.

Acknowledgments

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