Theoretical proof of acoustic source property single definition in liquid and solid

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The problem of acoustic waves property source single definition consists in unknown density and elastic modules of the medium. The Green's function definition is difficult as well as the real source isn't point and has the time dependence. In the early author's papers Green's function had been defined by using Levitan's polynomials. In the liquid or solid medium it is necessary for that to know only the first eigen values and shear or P-wave velocity at the first step from the free surface. It provides to calculate amplitudes of first mode of SH waves and the limits of lacunas in the spectrum. The time function of not moving source F is defined from the rolling up equation with the known displacement and Green's function. That equation is solved by direct and inverse fast Fourie transform. The further definition of source's location from the direct task for P or S body waves with defined velocity isn't hard by using modern finite element method. The rounding of the fronts gives the location of the source.

1 Direct task

Let's consider at first the case of solid medium and the direct task.

In case of layered medium with laterally homogeneous we have the following equation for SH-waves [1,4]:

$$\mu(z)\left[\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}-\frac{u}{r^2}\right]+\mu(z)\frac{\partial^2 u}{\partial z^2}+\frac{\partial \mu}{\partial z}\frac{\partial u}{\partial z}=\rho(z)\frac{\partial^2 u}{\partial t^2}+f \qquad (1)$$

At the free surface we have:

$$f=\overline{f}(z,r,t)+u_0(r,z)\delta'(t)+u_1(r,z)\delta(t), \qquad (2)$$

where $u\big|_{t<0}=0,\ \bar{f}\big|_{t<0}=0,$ [2]

the boundary condition: $\sigma_z\big|_{z=0}=0;\ \mu(z)\dfrac{\partial u}{\partial z}\Big|_{z=0}=0,$

and so $\dfrac{\partial u}{\partial z}\Big|_{z=0}=0.$ (3)

For the all axis the boundary conditions (4):

$$U\big|_{z=z_i+\varepsilon}=U\big|_{z=z_i-\varepsilon};\ \mu(z)\frac{\partial U}{\partial z}\bigg|_{z=z_i+\varepsilon}=\mu(z)\frac{\partial U}{\partial z}\bigg|_{z=z_i-\varepsilon} \qquad (4)$$

The condition of continuum of displacement and σ_z component stress tensor at the boundaries coal- rock, rock-rock ($z=z_i$)).

Where z- is vertical coordinate. Density $\rho(z)$ and shear modulus $\mu(z)$ are connected with the shear wave velocity by the formula $V_s=\sqrt{\dfrac{\mu}{\rho}}$.

The solution of the direct task is so [4-7]:

$$u(z,r,\varphi,t)=f*2i\sum_{n=1}^{N}\int_0^\infty\frac{1}{m_n^-}\psi_n^-(z_1,\eta_n)\psi_n^-(z_2,\eta_n)J_1(k_n\cdot r)E^{-i\omega t}\frac{\partial\omega}{2\pi}+f*G_{continuum} \qquad (5)$$

Where m_n^- – norms values of eigen functions $\psi_n^-(z,\eta_n);\ k_n$ – wave vector ($k_n=k_n^{'}+ik_n^{''}$); ω – circle frequency; J_1 – Bessel's function of the first order , $*$ – denominates rolling up.
Where

$$\psi^{\pm}(z_1,\lambda)=\sqrt{\frac{P(z_1,\lambda)}{P(\lambda)}}EXP\left\{\pm i\sqrt{R(\lambda)}\int_0^{z_1}\frac{\partial u}{P(u,\lambda)}\right\} \qquad (6)$$

$R(\lambda)=\lambda\prod_{k=1}^{N}(\lambda-\alpha_k)(\lambda-\beta_k);\ \alpha_k u\beta_k$ – boundaries of lacunas in spectrum , λ – eigen value of the Sturm-Liouville operator, $\lambda=\omega^2/V_s^2(0)-k^2$;

$$P(z_1,\lambda)=\prod_{k=1}^{N}(\lambda-\xi_k(z_1))\xi_k(0)=\xi_k,S(\lambda)=(\lambda-\eta_0)\prod_{k=1}^{N}(\lambda-\eta_k(0)) \qquad (7)$$

The elements $\xi(\lambda),\varsigma(\lambda),\eta(\lambda)$ of spectral matrix function $RM(\lambda)$ are defined by real polynomials $P(z_1,\lambda),S(\lambda)$ and function $R(\lambda)$, introduced by B.M. Levitan [8]:

$$\frac{\partial\xi}{\partial\lambda}=\frac{1}{2\pi}\cdot\frac{P(z_1,\lambda)}{\pm\sqrt{R(\lambda)}},\quad\frac{\partial\varsigma}{\partial\lambda}=\frac{1}{2\pi}\cdot\frac{S(\lambda)}{\pm\sqrt{R(\lambda)}},$$

$$\frac{\partial\eta}{\partial\lambda}=\frac{1}{2\pi}\cdot\frac{Q(z_1,\lambda)}{\pm\sqrt{R(\lambda)}} \qquad (8)$$

$$Q(z_1,\lambda)=P(z_1,\lambda)\cdot\sum_{k=1}^{N}\frac{\pm\sqrt{-R(\xi_K(z_1))}}{(\lambda-\xi_k(z_1))\cdot P^{'}(\xi_K(z_1))}. \qquad (9)$$

$\xi_j(z)=\left[\dfrac{\omega^2}{V_s^2(z)}-K_J^2\right]$ – vertical wave number in the second degree.

The general linear inverse problem has been developed by R.A. Wiggins [14]. The main requirements to the measuring velocity are described in [10]. Approach to analysis of dispersive surface waves is described in [11,12]. Separation of waves is known from the work [13].

2 Inverse task

We solve the following inverse task for SH wave equation: $u(z,r,\varphi,t)$ is known for minimum 4 receivers and 1 of them is on the depth $z=h$, 3 receivers are not on one line; It is necessary to find the domain of non point source (z,r,φ) and it's time function $f(z,r,\varphi,t)$.

The steps of solving the task of finding the properties of source are following:

1. Approximate definition of horizontal coordinates of the source and direction to it by using body waves and as minimum 4 receivers. 3 receivers are not belongs to one line, forth lays at the depth $z=h$.
2. Using the results of point 1 we define the real wave distance ΔR for neighbouring receivers.
Then we find dispersive curves for SH waves by spectral time analysis.
3. We must define $V_s(h)$ from point 1 for two receivers at the depth $z=h$, $V_s(0)$ for two receivers at the depth $z=0$ and know $\lambda_1=\omega^2/V_s^2(0)-k_1{}^2$ from dispersive curves obtained from point 2.
4. By given $V_s(h)$ and $\lambda_1=\omega^2/V_s^2(0)-k_1{}^2$ we calculate $A_1(\lambda_1)$ -amplitude of the first normal wave, α_1,β_1- boundaries of lacunas in the spectrum by the author's algorithm [4-7].
5. Solving the inverse task and building the vertical seismic cutting $V_s(z)$ when $\rho(z)=const$ by the analytical continuing $V_s(z)$ function by the author's algorithm [4-7].
6. We can use any method of solving the direct task (for example finite element method by the edition of Comsol Multiphysics) for the receivers which become point sources. Their type is the following

$$\omega_\varepsilon=C_\varepsilon\cdot\varepsilon\cdot\int\exp(-1/(1-|\xi|^2))d\xi, \tag{10}$$

where $\xi=\varepsilon/t$ and $\omega_\varepsilon\to\delta$ when $\varepsilon\to0$.

C_ε - must be calculated to obtain $\omega_\varepsilon=1$ [2].

The receivers start radiation in the inverse order in accordance with the time delays of body waves first arrivals. As the result we'll find the coordinates of the point source.

7. Then we solve the rolling up equation SH surface or channel waves for every receiver by direct and inverse Fourie transformer and obtain the function $f(z_i,r_i,t)=f(t)+\varphi(z_i,r_i,t)$, where $\varphi(z_i,r_i,t)$ define changes caused by non point source which gives the own time function.
8. Solving the rolling up equation body waves as well as $f(z_i,r_i,t)$ is known from point 7 and define Green's function for body waves.
9. Calculate rolling up $\Delta\varphi(t)*g(t)$ for receivers with coordinates (z_i,r_i).
10. Solving the direct task with finite element method for sources obtained in point 9. As the result we have the domain of non point source.

Let's proof all this steps.

Step 1.

The single definition of horizontal coordinates of the source and direction to it follows from the unique point of intersection in horizontal plane two hiperboles with the equation for the one's:

$$\frac{x^2}{\left(\frac{V_s(0)\Delta t}{2}\right)^2}-\frac{y^2}{l^2-\left(\frac{Vs(0)\Delta t}{2}\right)^2}=1,$$
$$\frac{(x')^2}{\left(\frac{V_s(0)\Delta t}{2}\right)^2}-\frac{(y')^2}{l^2-\left(\frac{Vs(0)\Delta t}{2}\right)^2}=1 \tag{11}$$

where $2l$ -is the distance between the neighbouring receivers, Δt - is the time delay, $V_s(0)-$is the speed at the depth $z=0$, x',y'- coordinates system for the second two receivers shifted at the given angle $\gamma\neq0$.

Step 2

$\Delta R=R_{i+1}-R_i$, where distance is defined by step 1. Then we find dispersive curves for SH waves by spectral time analysis and define the phase velocities. This method is well known.

Step3

Calculation first eigen value $\lambda_1=\omega^2/V_s^2(0)-k_1{}^2$ from the first mode dispersive curves of SH waves.

Step4

The method of definition the lacunas size in the Love waves spectrum [4-6]

Approving

If $A_1>>A_k,(k=2,N)$, than definition α_1,β_1, A_1 is reached by the known $V_s(h),\lambda_1$ from three equations:

$$\begin{aligned}&Q^2+R=0,\\&\alpha_1+\beta_1-2\xi_1(h)=0,\\&\sqrt{R(\lambda_1)}=P(\lambda_1)/A_1\end{aligned} \tag{12}$$

If k=1, known A_1,λ_1, than definition $V_s(h),\alpha_1,\beta_1$ is produced from the system of equations given above (12) with the condition:

$$V_s(h)=V_s(0)+\frac{\partial V(0)}{\partial z}h+o(h^2). \tag{13}$$

If known $A_1,\lambda_1,V_s(h)$ than α_1,β_1 can be defined from equations:

$$\begin{aligned}&Q^2+R=0,\\&\sqrt{R(\lambda_1)}=P(\lambda_1)/A_1\end{aligned} \tag{14}$$

The values of α_1 and β_1 are calculated in this way:

$$\beta_1^2(-\xi_1+\lambda_1)+\beta_1\left[\xi_1^2+\frac{(\lambda_1-\xi_1(h))^2}{A_1^2\lambda_1}-\lambda_1^2-\frac{(\lambda_1-\xi_1(h))^2}{A_1^2\xi_1}\right]+$$
$$+\frac{(\lambda_1-\xi_1(h))^2}{A_1^2\xi_1}\lambda_1-\lambda_1\xi_1^2-\frac{(\lambda_1-\xi_1(h))^2}{A_1^2\lambda_1}\xi_1+\lambda_1^2\xi_1=0. \tag{15}$$

$$\alpha_1=\left[\frac{(\lambda_1-\xi_1(h))^2}{A_1^2\cdot\lambda_1}-\lambda_1^2+\lambda_1\cdot\beta_1\right]/(-\lambda_1+\beta_1) \tag{16}$$

where A_1 - the amplitudes of first mode normal Love wave;

where $\beta_1 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ (17)

the sign before root must be thus as to have β_1 non negative. For reaching the stability of calculations in the dividers of formulas (15-17) ε should be added.
From accounting above it is seen that if the amplitudes set is the following: $A_1, 0, 0 \ldots 0$, than α_1, β_1 are calculated exactly, but if all members A_k of the set are not zero, than α_1', β_1' are calculated approximately. In that case the value $A_1' = \dfrac{A_1}{\prod_{k=2}^{N}(\lambda_1 - \xi_k)}$ should be taken as the first mode amplitudes for implementing the condition:

$$\frac{P(\lambda_1)}{A_1} = \frac{P'(\lambda_1)}{A_1'}, \qquad (18)$$

Where $P'(\lambda_1) = (\lambda_1 - \xi_1)$, $P(\lambda_1) = \prod_{k=1}^{N}(\lambda_1 - \xi_k)$ (19)

Step5
After separation of the variables and substitution by Tricomy's method we have the following operator:

$$L(U) = \frac{\partial^2 U}{\partial z^2} + \left(-d(z) + \frac{\omega^2}{V_s^2(z)} - k^2\right)U = 0, \qquad (20)$$

Where

$$d(z) = -\frac{1}{4}\left(\frac{\partial \mu}{\partial z}/\mu\right)^2 + \frac{1}{2}\frac{\partial^2 \mu}{\partial z^2}/\mu \qquad (21)$$

Green's function of the Sturm-Liouville equation is so [3]:

$$G(z_1, z_2, Z) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{P(\lambda)}{(\lambda - Z)\sqrt{R(\lambda)}}\psi^{\pm}(z_1,\lambda)\psi^{\mp}(z_2,\lambda)\partial\lambda \quad (22)$$

Where

$$\psi^{\pm}(z_1,\lambda) = \sqrt{\frac{P(z_1,\lambda)}{P(\lambda)}}EXP\left\{\pm i\sqrt{R(\lambda)}\int_0^{z_1}\frac{\partial u}{P(u,\lambda)}\right\} \quad (23)$$

From the consideration the Weil's functions follows: $S(\eta_K) = 0$ is connected with channel and surface waves (residual spectrum), but zeroes of the root

$$\pm\sqrt{P(z_1,\lambda)S(\lambda) - R(\lambda)} = Q(z_1,\lambda) \qquad (24)$$

connected with the side body waves (continuum spectrum),
$Q(z,\eta_k) = P(\eta_k)\mu(z)\partial W(z,\lambda)/\partial\lambda\big|_{\lambda=\eta_k} = \sqrt{-R(\eta_k)}$ (25).
That equivalence the condition of solving the inverse Sturm-Liouville task (24), in another way could be found $\mu(z)$, where $z \in [0, l]$, that condition (25) is true.

It is seen, that $\dfrac{\partial Q(z,\eta_k)}{\partial z} = 0$ and $W(0,\lambda) \neq 0$ for real λ.

$$q(z_1) = \sum_{k=1}^{N} \lambda_k - \xi_k(z_1) + d(z_1). \qquad (26)$$

The method consists of analytical continuum of the wave velocity function $V_s(z)$ by using signs and Taylor's formula :

$$q(t) = \sum_{k=1}^{N}\alpha_k + \beta_k - 2\xi_k(t) \qquad (27)$$

for condition

$$\rho(z) = const, \text{ or } (\sqrt{\mu})''_z / \sqrt{\mu} = o(1). \qquad (28)$$

Author's Fortran program «D3» allows to define the vertical seismic cutting and the limits of it's possible changes by the condition $\xi_1(z_1) = \alpha_1, \beta_1$. (29)
From works of B.M.Levitan [8] and analysis of the real amplitudes of Love waves a_j it directly follows that all points $\xi_j(t)$ satisfy the difference equation:

$$\frac{\partial \xi_j}{\partial t} = \frac{\pm 2a_j\sqrt{-R(\xi_j(t))}}{\prod_{k \neq j}^{N}(\xi_K - \xi_j(t))}, j = 1....N, \qquad (30)$$

$$R(\xi) = \xi\prod_{k=1}^{N}(\xi - \alpha_k)(\xi - \beta_k), \qquad (31)$$

$$\xi_k \in [\alpha_k, \beta_k], \qquad (32)$$

where $\xi_j(z) = \left[\dfrac{\omega^2}{V_s^2(z)} - K_J^2\right]$ – variable vertical wave number in the second degree, t- shift.
If N=1, than divider is equal to 1. Point ξ_j always belongs to the lacunas that is provided by changing the sign before root. In the opposite case the frequency is missed.
If N=1, than the following equation [5,6] can be used (33):

$$\frac{\partial^2 V_s(t)}{\partial t^2} = \frac{-V_s^2(t)\sum_{i=1}^{M}(\lambda_1(i) - (2\pi F(i)/V(i))^2 - (\alpha_1(i) + \beta_1(i))) - \sum_{i=1}^{M}(2\pi F(i))^2}{MV_s(t)}$$

Where:
- $\alpha_1(i)$ and $\beta_1(i)$ – boundaries of the first lacunas in the Love wave's spectrum;
- F(i)- frequency of Love waves;
- i- frequency's number after spectral time analysis of the first Love wave mode;
- V(i)- phase velocity of Love wave.

If λ_k <<1, and exactly known $V_s(h)$, than it is enough to use only one first mode. The mistake of calculation $R(\lambda_k)$ function has the order $o(\lambda_k^4)$.
Using not more than three shifts on the frequency and knowing $V_s(h)$ also available to use only the first mode. It's followed from interpolation theory.
Step6
Solving the direct task for the equations with approximation continuum medium by several layers $(1 \leq i \leq N)$:

$$\frac{\partial^2 u}{\partial t^2} - V_s^2(z_1)\Delta u = \omega_\varepsilon(t - \tau_i); \text{ for } z_i = z_{rec}$$

$$\frac{\partial^2 u}{\partial t^2} - V_s^2(z_1)\Delta u = 0; \qquad (34)$$

with the boundary condition on free surface

$$\frac{\partial u}{\partial z}\Big|_{z=0} = 0. \qquad (35)$$

For the inside boundaries we have:

$$U\big|_{z=z_i+\varepsilon} = U\big|_{z=z_i-\varepsilon}; \mu(z)\frac{\partial U}{\partial z}\Big|_{z=z_i+\varepsilon} = \mu(z)\frac{\partial U}{\partial z}\Big|_{z=z_i-\varepsilon}. \qquad (36)$$

The point sources start in the inverse order. The method of solving the system of equations may be finite element method. As the result we'll find the coordinates of the point source.

Step 7

We have the following rolling up equation SH surface or channel waves:

$$u(z_i, r_i, t) = f(z_i, r_i, t) * G(z_{sorc}, z_{rec}, r_{rec}, t) \qquad (37),$$

where G- is Green's function for residual spectrum.

We solve this equation by direct and inverse Fourie transformer by the method described in [9]:

$$F(u) = F(f)\cdot F(G),\ F(f) = \frac{F(u)}{F(G)},$$

$$f = F^{-1}\left(\frac{F(u)}{F(G)+\varepsilon}\right) \qquad (38),$$

where $\varepsilon -$ is the regularization parameter.

Obtained function $f(z_i, r_i, t) = f(t) + \varphi(z_i, r_i, t)$ (39), where $\varphi(z_i, r_i, t)$ defines changes caused non point source which gives the own time function.

Step8

Solving the rolling up equation for body waves:

$$u(z_i, r_i, t) = f(z_i, r_i, t) * g(z_{sorc}, z_{rec}, r_{rec}, t) \qquad (40)$$

and define Green's function g for body waves by above described method.

Step9

Calculate rolling up $\Delta\varphi(t) * g(t)$ for receivers with coordinates (z_i, r_i).

We have used the rolling up property difference:

$$(f(z,r,t) * g(z,r,t))'_r = (f'_r * g) \cong (\frac{\Delta\varphi}{\Delta r} * g)(t) \qquad (41)$$

As well as the medium is non homogeneous and source isn't point $\Delta\varphi_{source}(t) \neq \Delta\varphi_{receiver}(t)$.

For point source $\Delta\varphi_{source}(t) = 0$, for oscillation sphere in homogeneous medium $\Delta\varphi_{source}(t) = \Delta\varphi_{receiver}(t)$. The source is point, but it's function isn't zero in that interval $t \in [0, \tau]$, where

$$\tau = \frac{D_{\max}}{V_s(h)} \qquad (42)$$

and $D_{\max}$ - is maximum size of the source domain. It is the result of using the interdependence theorem in acoustics.

Step 10.

Repeat step 6 for the sources body waves obtained in step 9. As the result we have the domain of non point source composed from the wave fronts.

The source size is defined from equal

$$D = \tau_\varphi \cdot V_s(h), \qquad (43)$$

where τ_φ - time of source radiation for the angle φ.

The liquid medium.

The equation is the following [1]:

$$\frac{d}{dt}\left[\frac{d}{dt}\left(\frac{1}{\rho v^2}\frac{dp}{dt}\right) - div\left(\frac{\nabla p}{\rho}\right)\right] + 2\left(\frac{dV_0}{dz}\nabla\right)\left(\frac{1}{\rho}\frac{dp}{dz}\right) = 0, \qquad (44)$$

Where $|V_0| << V$, V_0 -the speed of moving the fluid.

At the free surface we have $p + p_0 = 0$, initial conditions: $P\big|_{t<0} = 0, \frac{\partial p}{\partial t}\Big|_{t<0} = 0$ (45)

The task as a result is reduced to the following (46) Helmgholts equation [1]:

$$\frac{\partial^2}{\partial z^2}\psi + \left[\frac{\omega^2}{V^2}\gamma^2 - k^2 + \frac{1}{2\rho\gamma^2}\frac{\partial^2}{\partial z^2}(\rho\gamma^2) - \frac{3}{4}\left(\frac{1}{\rho\gamma^2}\frac{\partial}{\partial z}\rho\gamma^2\right)^2\right]\psi = 0$$

Where $\gamma = 1 - kV_0/\omega$,

$$\psi(z,k,\omega) = p(z,k,\omega)/\gamma(z,k,\omega)\sqrt{\rho(z)} \qquad (47)$$

This task has the same way of solving, but we must find V_0. The direct task is connected with solving the equation (44) by finite element method

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