Theoretical proof of acoustic source property single definition in liquid and solid

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1 Direct task

Let’s consider at first the case of solid medium and the direct task.

\[ \mu(z) \left[ \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} - \frac{u}{r^2} \right] + \mu(z) \frac{\partial^2 u}{\partial z^2} + \frac{\partial \mu}{\partial z} \frac{\partial u}{\partial z} = \rho(z) \frac{\partial^2 u}{\partial t^2} + f \]  

(1)

At the free surface we have:

\[ f = \tilde{f}(z,r,t) + u_0(r,z)\delta(t) + u_1(r,z)\delta(t), \]

where \( u|_{r=0} = 0, \tilde{f}|_{r=0} = 0 \),

the boundary condition: \( \sigma_z|_{z=0} = 0; \mu(z) \frac{\partial u}{\partial z} |_{z=0} = 0 \),

and so

\[ \frac{\partial u}{\partial z} |_{z=0} = 0. \]

(3)

For all the axis the boundary conditions (4):

\[ U|_{z=\pm \infty} = \frac{\partial U}{\partial z} |_{z=\pm \infty} = \frac{\partial U}{\partial z} |_{z=\pm \infty} = 0 \]

where \( m_n \) - norms values of eigen functions \( \psi_n(z,\eta_j) \), \( k_n \) - wave vector \( k_n = k_i + ik_n^p \);

\( \omega \) - circle frequency; \( J_1 \) - Bessel’s function of the first order; * - denominates rolling up.

Where

\[ \psi^p(z,\lambda) = \sqrt{P(z,\lambda)} \exp \left\{ \pm i \sqrt{R(\lambda)} \int_0^z \frac{\partial u}{\partial z} \right\} \]

(6)

\[ R(\lambda) = \lambda \prod_{k=1}^{N} (\lambda - \alpha_k)(\lambda - \beta_k) \]

\( \alpha_k, \beta_k \) - boundaries of lacunas in spectrum , \( \lambda \) - eigen value of the Sturm-Liouville operator, \( \lambda = \omega^2 / V_s^2 (0) - k^2 \);

\[ H_z(\lambda) = \sum_{k=1}^{N} \left( \lambda - \xi_k(\lambda) \right) \xi_k(0) = \xi(\lambda - \eta) \sum_{k=1}^{N} \left( \lambda - \eta \right) \]

The elements \( \xi(\lambda), \zeta(\lambda), \eta(\lambda) \) of spectral matrix function \( RM(\lambda) \) are defined by real polynomials

In case of layered medium with laterally homogeneous we have the following equation for SH-waves [1,4]:

The condition of continuum of displacement and \( \sigma_z \) component stress tensor at the boundaries coal-rock, rock-rock \( z = z_j \).

Where \( z \) - is vertical coordinate. Density \( \rho(z) \) and shear modulus \( \mu(z) \) are connected with the shear wave velocity by the formula

\[ V_s = \sqrt{\frac{\mu}{\rho}}. \]

The solution of the direct task is so [4-7]:

\[ P(z_1,\lambda) \text{ S}(\lambda) \text{ and function } R(\lambda), \text{ introduced by B.M. Levinan [8]}: \]

\[ \frac{\partial \xi}{\partial \lambda} = \frac{1}{2\pi} \sqrt{R'(\lambda)}, \frac{\partial \eta}{\partial \lambda} = \frac{1}{2\pi} \sqrt{R'(\lambda)}, \]

(8)

\[ Q(z_1,\lambda)|_{\lambda=0} = \sum_{k=1}^{N} \left( \lambda - \xi_k(\lambda) \right) \text{ and } \frac{\partial \xi_k}{\partial \lambda} = \frac{1}{2\pi} \sqrt{R'(\lambda)} \]

(9)

The general linear inverse problem has been developed by R.A. Wiggins [14]. The main requirements to the measuring velocity are described in [10]. Approach to analysis of dispersive surface waves is described in [11,12]. Separation of waves is known from the work [13].
2 Inverse task

We solve the following inverse task for SH wave equation: 

\[ u(z, r, \varphi, t) \]

is known for minimum 4 receivers and 1 of them is on the depth \( z = h \), 3 receivers are not on one line; 
It is necessary to find the domain of non point source \((z, r, \varphi)\) and it's time function \(f(z, r, \varphi, t)\).

The steps of solving the task of finding the properties of source are following:
1. Approximate definition of horizontal coordinates of the source and direction to it by using body waves and as minimum 4 receivers. 3 receivers are not belongs to one line, forth lays at the depth \( z = h \).
2. Using the results of point 1 we define the real wave distance \( \Delta R \) for neighbouring receivers. Then we find dispersive curves for SH waves by spectral time analysis.
3. We must define \( V_s(h) \) from point 1 for two receivers at the depth \( z = h \). \( V_s(0) \) for two receivers at the depth \( z = 0 \) and know \( \lambda_i = \omega^2 / \beta_i(0) - k_i^2 \) from dispersive curves obtained from point 2.
4. By given \( V_s(h) \) and \( \lambda_i = \omega^2 / \beta_i(0) - k_i^2 \) we calculate \( A_i(\lambda_i) \) - amplitude of the first normal wave, \( \alpha_i, \beta_i \) - boundaries of lacunas in the spectrum by the author’s algorithm [4-7].
5. Solving the inverse task and building the vertical seismic cutting \( V_s(z) \) when \( \rho(z) = const \) by the analytical continuing \( V_s(z) \) function by the author’s algorithm [4-7].
6. We can use any method of solving the direct task (for example finite element method by the edition of Comsol Multiphysics) for the receivers which become point sources. Their type is the following

\[ \omega_{\varphi} = C_{\varphi} \cdot e \cdot \int \exp(-1/(1-|\xi^2|))d\xi, \]  

where \( \xi = \varphi / t \) and \( \omega_{\varphi} \rightarrow \delta \) when \( e \rightarrow 0 \).

The receivers start radiation in the inverse order in accordance with the time delays of body waves first arrivals. As the result we’ll find the coordinates of the point source.
7. Then we solve the rolling up equation SH surface or channel waves for every receiver by direct and inverse Fourier transform and obtain the function \( f(z, r, t) = f(t) + \varphi(z, r, t) \), where \( \varphi(z, r, t) \) define changes caused by non point source which gives the own time function.
8. When solving the rolling up equation body waves as well as \( f(z, r, t) \) is known from point 7 and define Green’s function for body waves.
9. Calculate rolling up \( \Delta \varphi(t) * g(t) \) for receivers with coordinates \((z_i, r_i)\).
10. Solving the direct task with finite element method for sources obtained in point 9. As the result we have the domain of non point source.

Let’s proof all this steps.

Step 1.
The single definition of horizontal coordinates of the source and direction to it follows from the unique point of intersection in horizontal plane two hiperboles with the equation for the one’s:

\[ \frac{x^2}{2} - \frac{y^2}{2} = 1 \]

where 2l - is the distance between the neighbouring receivers, \( \Delta t \) - is the time delay, \( V_s(0) \) is the speed at the depth \( z = 0 \), \( x', y' \) - coordinates system for the second two receivers shifted at the given angle \( \gamma \neq 0 \).

Step 2
\[ \Delta R = R_{i+1} - R_i, \] where distance is defined by step 1.

Then we find dispersive curves for SH waves by spectral time analysis and define the phase velocities. This method is well known.

Step 3
Calculation first eigen value \( \lambda_1 = \omega^2 / \beta_1(0) - k_1^2 \) from the first mode dispersive curves of SH waves.

Step 4
The method of definition the lacunas size in the Love waves spectrum [4-6]

\[ \text{Approving} \]

If \( A_i \gg A_k, (k = 2, N) \), than definition \( \alpha_i, \beta_i, A_i \) is reached by the known \( V_s(h), \lambda_i \) from three equations:

\[ \begin{aligned}
Q^2 + R &= 0, \\
\alpha_i + \beta_i - 2\xi_i(h) &= 0, \\
\sqrt{R(\lambda_i)} &= P(\lambda_i) / A_i
\end{aligned} \]

If \( k=1 \), known \( A_1, \lambda_1 \), than definition \( V_s(h), \alpha_i, \beta_i \) is produced from the system of equations given above (12) with the condition:

\[ V_s(h) = V_s(0) + \frac{\partial V(0)}{\partial h} h + o(h^2). \]

If known \( A_1, \lambda_1, V_s(h) \) than \( \alpha_i, \beta_i \) can be defined from equations:

\[ \begin{aligned}
Q^2 + R &= 0, \\
\sqrt{R(\lambda_i)} &= P(\lambda_i) / A_i
\end{aligned} \]

The values of \( \alpha_i \) and \( \beta_i \) are calculated in this way:

\[ \begin{aligned}
\beta_i\xi_i(-\xi_i + \lambda_i) + \beta_i\frac{\xi_i^2 + (\lambda_i - \xi_i(h))^2}{\lambda_i A_i} &= (-\lambda_i + \xi_i(h))^2, \\
\xi_i\left(-\frac{(\lambda_i - \xi_i(h))^2}{\lambda_i A_i} - \frac{(\lambda_i - \xi_i(h))^2}{\lambda_i A_i} - \xi_i\right) &= 0.
\end{aligned} \]

\[ \alpha_i = \frac{(-\lambda_i - \xi_i(h))^2}{\lambda_i A_i} - \lambda_i - \lambda_1 - \beta_i \]
where \( A_1 \) - the amplitudes of first mode normal Love wave;

where \( \beta_1 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \) (17)

the sign before root must be thus as to have \( \beta_1 \) non negative. For reaching the stability of calculations in the dividers of formulas (15-17) \( \varepsilon \) should be added.

From accounting above it is seen that if the amplitudes set is the following: \( A_1, 0, 0 \ldots 0\), then \( \alpha_1, \beta_1 \) are calculated exactly, but if all members \( A_k \) of the set are not zero, than \( \alpha_1', \beta_1' \) are calculated approximately. In that case the value \( A_1' = \frac{N}{\sum_{k=2}^{N} (\lambda_k - \xi_k)} \) should be taken as the first mode amplitudes for implementing the condition:

\[
P(\lambda_1') = P(\lambda_1) \]

Where \( P(\lambda_1') = (\lambda_1' - \xi_k) \), \( P(\lambda_1) = \prod_{k=1}^{N} (\lambda_1 - \xi_k) \) (19)

Step5

After separation of the variables and substitution by Tricomy’s method we have the following operator:

\[
L(U) = \frac{\partial^2 U}{\partial z^2} + (-d(z) + \frac{\omega^2}{V_s^2(z)}) - k^2 U = 0,
\]

Where

\[
d(z) = -\frac{1}{4} \frac{\partial \mu}{\partial z} + \frac{1}{2} \frac{\partial^2 \mu}{\partial z^2} \] (21)

Green’s function of the Sturm-Liouville equation is so [3]:

\[
G(z_1, z_2, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P(\lambda) \psi^*(z_1, \lambda) \psi(z_2, \lambda) d\lambda
\]

(22)

Where

\[
\psi^*(z_1, \lambda) = \frac{P(z_1, \lambda)}{P(\lambda)} \exp \left\{ \pm i \sqrt{R(\lambda)} \int_0^z \frac{\partial \mu}{\partial \mu} \right\}
\]

(23)

From the consideration the Weil’s functions follows: \( S(\eta_k) = 0 \) is connected with channel and surface waves (residual spectrum), but zeroes of the root

\[ \pm \sqrt{P(z_1, \lambda) S(\lambda) - R(\lambda) = Q(z_1, \lambda)} \] (24)

(24) connected with the side body waves (continuum spectrum), \( Q(z, \eta_i) = P(\eta_i) \mu(z) \partial W(z, \lambda)/\partial \lambda \bigg|_{\eta_i} = -\sqrt{R(\eta_i)} \) (25).

That equivalence the condition of solving the inverse Sturm-Liouville task (24), in another way could be found \( \mu(z) \), where \( z \in [0, L] \) that condition (25) is true.

It is seen, that \( \partial Q(z, \eta_i)/\partial z = 0 \) and \( W(0, \lambda) \neq 0 \) for real \( \lambda \). \( q(z_1) = \sum_{i=1}^{N} \lambda_k - \xi_k(z_1) + d(z_1) \). (26)

The method consists of analytical continuum of the wave velocity function \( V_s(z) \) by using signs and Taylor’s formula:

\[
q(t) = \sum_{i=1}^{N} \alpha_k + \beta_k - 2 \xi_k(t)
\]

where

\[
\rho(z) = \text{const}, \text{or } \frac{\sqrt{\mu}^n}{\sqrt{\mu} = o(1)}.
\]

Author’s Fortran program «D3» allows to define the vertical seismic cutting and the limits of it’s possible changes by the condition \( \xi_k(z_1) = \alpha_1, \beta_1 \). (29)

From works of B.M. Levitan [8] and analysis of the real amplitudes of Love waves \( a_i \) it directly follows that all points \( \xi_j(t) \) satisfy the difference equation:

\[
\frac{\partial \xi_j(t)}{\partial t} = \pm 2a_j \frac{-R(\xi_j(t))}{\prod_{k=1}^{N} (\xi_k - \xi_j(t))}, j = 1 \ldots N,
\]

(30)

where \( \xi_j(z) = \sqrt{\omega^2 V_s^2(z) - K_j^2} \) - variable vertical wave number in the second degree, t- shift.

If \( N = 1 \), than divider is equal to 1. Point \( \xi_j \) always belongs to the lacunas that is provided by changing the sign before root. In the opposite case the frequency is missed.

If \( N = 1 \), than the following equation [5,6] can be used (33):

\[
\frac{\partial V_s(t)}{\partial t} = -V_s(t) \sum_{i=1}^{N} (\lambda_i(t) - (2 \pi f(i) / V_s(t) - (\alpha_i(t) + \beta_i(t))) \sum_{i=1}^{N} (2 \pi f(i))^2
\]

(34)

Where:

- \( \alpha_i(t) \) and \( \beta_i(t) \) - boundaries of the first lacunas in the Love wave’s spectrum;
- \( f(i) \) - frequency of Love waves;
- \( k \) - frequency’s number after spectral time analysis of the first Love wave mode;
- \( V(t) \) - phase velocity of Love wave.

If \( \lambda_k < 1 \), and exactly known \( V_s(h) \), than it is enough to use only one first mode. The mistake of calculation \( R(\lambda_k) \) function has the order \( o(\lambda_k^2) \).

Using not more than three shifts on the frequency and knowing \( V_s(h) \) also available to use only the first mode. It’s followed from interpolation theory.

Step6

Solving the direct task for the equations with approximation continuum medium by several layers \( 1 \leq i \leq N \):

\[
\frac{\partial^2 u}{\partial t^2} - V_s^2(z_1) \Delta u = \omega_s(t - \tau_i) \]

for \( z_i = z_{rec} \)

\[
\frac{\partial^2 u}{\partial t^2} - V_s^2(z_1) \Delta u = 0;
\]

(34)
with the boundary condition on free surface
\[
\frac{\partial u}{\partial z}|_{z=0} = 0.
\]  
(35)

For the inside boundaries we have:
\[
U|_{z=z_{se}} = U|_{z=z_{ec}} ; \mu(z) \frac{\partial U}{\partial z}|_{z=z_{se}} = \mu(z) \frac{\partial U}{\partial z}|_{z=z_{ec}}.
\]  
(36)

The point sources start in the inverse order. The method of solving the system of equations may be finite element method. As the result we’ll find the coordinates of the point source.

Step 7

We have the following rolling up equation SH surface or channel waves:
\[
u(z, r, t) = \phi(z, r, t) \ast \Gamma(z, r, t),
\]
(37)
where \( \Gamma \) is Green’s function for residual spectrum. We solve this equation by direct and inverse Fourier transformer by the method described in [9]:
\[
F(u) = F(f) \cdot F(G),
\]
(38)

\[
f = F^{-1}\left(\frac{F(u)}{F(G)} + \varepsilon\right),
\]
where \( \varepsilon \) – is the regularization parameter.

Obtained function \( f(z, r, t) = f(t) + \phi(z, r, t) \) (39),
where \( \phi(z, r, t) \) defines changes caused non point source which gives the own time function.

Step 8

Solving the rolling up equation for body waves:
\[
u(z, r, t) = \phi(z, r, t) \ast \Gamma(z, r, t),
\]
(40)
and define Green’s function \( g \) for body waves by above described method.

Step 9

Calculate rolling up \( \Delta \phi(t) \ast g(t) \) for receivers with coordinates \( (z, r) \).

We have used the rolling up problem difference:
\[
(f(z, r, t) \ast g(z, r, t))_t = \frac{\partial}{\partial t} \phi(z, r, t) \ast g(t) \equiv \frac{\Delta \phi}{\Delta t} \ast g(t),
\]
(41)
As well as the medium is non homogeneous and source isn’t point \( \Delta \phi_{source}(t) \neq \Delta \phi_{receiver}(t) \).

For point source \( \Delta \phi_{source}(t) = 0 \), for oscillation sphere in homogeneous medium \( \Delta \phi_{source}(t) = \Delta \phi_{receiver}(t) \). The source is point, but it’s function isn’t zero in that interval \( t \in [0, \tau] \), where
\[
\tau = \frac{D_{max}}{V_s(h)}
\]
(42)
and \( D_{max} \) - is maximum size of the source domain. It is the result of using the interdependence theorem in acoustics.

Step 10

Repeat step 6 for the sources body waves obtained in step 9. As the result we have the domain of non point source composed from the wave fronts.

The source size is defined from equal
\[
D = \tau \ast V_s(h),
\]
(43)

where \( \tau \) - time of source radiation for the angle \( \varphi \).

The liquid medium.

The equation is the following [1]:
\[
d\left[ d\left( \frac{1}{\rho} \frac{\partial p}{\partial t} \right) - \frac{\nabla p}{\rho} \right] + \frac{dV_s}{dz} \left( \frac{1}{\rho} \frac{\partial p}{\partial z} \right) = 0,
\]
(44)

Where \( \left| \frac{dV_s}{dz} \right| < V_s \), \( V_s \)-the speed of moving the fluid.

At the free surface we have \( p + p_0 = 0 \), initial conditions:
\[
P|_{t=0} = 0, \frac{\partial p}{\partial t}|_{t=0} = 0
\]
(45)

The task as a result is reduced to the following (46) Helmholtz equation [1]:
\[
\frac{\partial^2 \psi}{\partial z^2} + \left[ \frac{a^2}{V_s^2} - k^2 \right] \psi + \frac{1}{2\rho^2} \frac{\partial^2 (\rho \psi^2)}{\partial z^2} \left( \frac{1}{\rho} \frac{\partial \psi}{\partial z} \right)^2 \psi = 0
\]
(47)

This task has the same way of solving, but we must find \( V_s \). The direct task is connected with solving the equation (44) by finite element method.

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References

Acoustics 08 Paris


